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OPEN AND DISTANCE LEARNING (ODL) PROGRAMMES

(FOR THOSE WHO JOINED THE PROGRAMMES FROM THE ACADEMIC YEAR 2023–2024)

M.Sc. Physics

Course Material

CLASSICAL MECHANICS AND RELATIVITY

Prepared

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Tirunelveli - 12

CLASSICAL MECHANICS AND RELATIVITY

UNIT I: PRINCIPLESOF CLASSICAL MECHANICS

Mechanics of a single particle – conservation laws for a particle – mechanics of a system of particles – conservation laws for a system of particles – constraints – holonomic & non-holonomic constraints – generalized coordinates–configuration space–transformation equations – principle of virtual work.

UNIT II: LAGRANGIAN FORMULATION

D'Alembert's principle – Lagrangian equations of motion for conservative systems – applications: (i) simple pendulum (ii) Atwood's machine – Lagrange's equations in presence of non- –Lagrangian for a charged particle moving in an Electromagnetic field.

UNIT III: HAMILTONIAN FORMULATION

Phase space – generalized momentum and cyclic coordinates – Hamiltonian function and conservation of energy – Hamilton's canonical equations of motion – applications: (i) one dimensional simple harmonic oscillator (ii)motion of particle in a central force field.

UNIT IV: SMALL OSCILLATIONS

Stable and unstable equilibrium –Formulation of the problem: Lagrange's equations of motion for small oscillations – Properties of T, Vand w– Normal co ordinates and normal frequencies of vibration– Free vibrations of a linear triatomic molecule.

UNIT V: RELATIVITY

Inertial and non-inertial frames – Lorentz transformation equations – length contraction and time dilation–relativistic addition of velocities – Einstein's mass-energy relation Minkowski's space–four vectors – position, velocity, momentum, acceleration and force in four vector notation and their transformations.

UNIT VI: PROFESSIONAL COMPONENTS

Expert Lectures, Online Seminars - Webinars on Industrial Interactions/Visits, Competitive Examinations, Employable and Communication Skill Enhancement, Social Accountability and Patriotism

TEXT BOOKS

1. H.Goldstein, *ClassicalMechanics*,3rdEdition,PearsonEdu. 2002.
2. J.C.Upadhyaya, *ClassicalMechanics*, Himalaya PublshingCo. New Delhi.
3. S.L.Gupta,V.Kumar,H.V.Sharma,ClassicalMechanics,PrakatiPrakashan, Meerut.
4. R. Resnick, *Introduction to Special Theory of Relativity*, Wiley Eastern, New Delhi, 1968.
5. N.C.RanaandP.S.Joag, ClassicalMechanics - TataMcGraw Hill, 2001

UNIT I: PRINCIPLES OF CLASSICAL MECHANICS

Mechanics of a single particle – conservation laws for a particle – mechanics of a system of particles – conservation laws for a system of particles – constraints – holonomic & non-holonomic constraints – generalized coordinates – configuration space – transformation equations – principle of virtual work.

1.1 Mechanics of a single particle : conservation laws

Newtonian mechanics to deduce conservation laws for a particle in motion. These laws tell under what conditions the mechanical quantities like linear momentum, angular momentum, energy etc. are constant in time.

Conservation of linear momentum:

If the force F is acting on a particle of mass m, then according to Newton's law of motion, we have

$$F = \frac{dp}{dt} = \frac{d(mv)}{dt}$$

Where p = mv is the linear momentum of the particle.

If the external force, acting on the particle is zero, then

$$\frac{dp}{dt} = \frac{d(mv)}{dt} = 0$$

Thus in absence of external force, the linear momentum of a particle is conserved. This is conservation theorem for a free particle.

Conservation of angular momentum:

The angular momentum of a particle P of a mass ma about a point O is

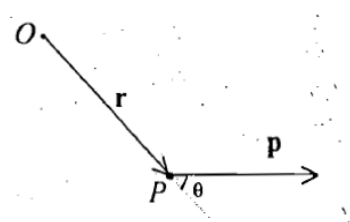

Fig. 1.1

$$J = r \times p \qquad \text{-----------------------(1)}$$

Where r is the position vector of the particle P and p = mv is its linear momentum.

If the force on the particle is F, then the moment of force of torque about O is defined as

$$\tau = r \times F \qquad \text{-----------------(2)}$$

Differentiate equation (1) with respect to t

$$\frac{dJ}{dt} = \frac{d(\mathrm{r} \times \mathrm{p})}{dt} = \frac{d\mathrm{r}}{dt} \times \mathrm{p} + \mathrm{r}\frac{d\mathrm{r}}{dt} \times \mathrm{p}\ \frac{d\mathrm{p}}{dt}$$

$$\frac{dJ}{dt} = \mathrm{r} \times F$$

$$\left(\frac{d\mathrm{r}}{dt} \times \mathrm{p} = \mathrm{v} \times \mathrm{mv}=0; \quad \mathrm{F} = \frac{dp}{dt}\right)$$

$$\tau = \frac{dJ}{dt} = \frac{d(\mathrm{r} \times \mathrm{p})}{dt} \text{- -------------(3)}$$

Thus the time rate of change of angular momentum of a particle is equal to the torque acting on it.

If the torque acting on the particle is zero, i.e., $\tau = 0$, then

$$\frac{dJ}{dt} = 0 \text{ or } \mathrm{J} = \text{constant}$$

Therefore the angular momentum of a particle is constant of motion in absence of external torque. This is the conservation theorem of angular momentum of a particle.

Conservation of energy:

(a) Work: work done by an external force F upon a particle in displacing from point 1 to point 2 is defined as

$$\mathrm{W}_{12} = \int_1^2 F \,.\, dr \quad \text{--------------(4)}$$

(b) Kinetic energy and Work – Energy theorem:

According to newton's second law F = m $\frac{dv}{dt}$ and hence

$$\mathrm{F} \,.\, \mathrm{dr} = \mathrm{m}\frac{dv}{dt} .\ \mathrm{dr}$$

$$= \mathrm{m}\frac{dv}{dt} .\ \mathrm{v\,dt} \qquad \left(\mathrm{dr} = \frac{dr}{dt}\, dt = v\,.\,dt\right)$$

$$= \mathrm{m}\frac{d(\frac{1}{2}v\cdot v)}{dt}\ \mathrm{dt}$$

$$= \mathrm{d}\left(\frac{1}{2}mv^2\right)$$

Therefore, equation (4) is

$$\mathrm{W}_{12} = \int_1^2 F \,.\, dr = \mathrm{d}\left(\frac{1}{2}mv^2\right)$$

$$= \frac{1}{2} m\, v_2^2 - \frac{1}{2} m\, v_1^2$$

Thus the work done by the force acting on the particle appears equal to the change in the kinetic energy. i.e.,

$$W_{12} = \int_1^2 F \,.\, dr = T_2 - T_1$$

This is known as **Work – Energy theorem**.

(c) Conservation of force and potential energy: If the force acting on a particle is such that, the work done along the closed path is zero. Then the force is called conservative force.

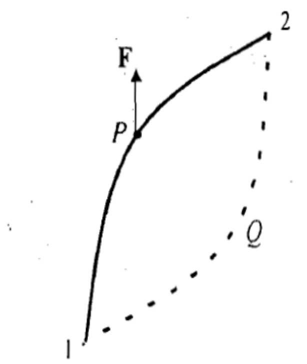

Fig. 1.2

$$P \int_1^2 F.dr = Q \int_1^2 F.dr$$

$$P \int_1^2 F.dr - Q \int_1^2 F.dr = 0$$

The total work done is zero. Because it is a closed path.

i.e., $\oint F \cdot dr = 0$

In closed path displacement (dr) also zero.

Conditions:

(i) Accoding to storke's theorem

$\oint_l F \,.\, dl = \iint_S ds\, (\nabla \times F)$

(ii) In this case ds $\neq 0$, $\nabla \times F = 0$

(iii) Curl F = 0 or

$\nabla \times F = 0$

(iv) F = - ∇v

$$\nabla \times \nabla v = 0$$

$$F = -\nabla v = -\left(\hat{\imath}\frac{\partial v}{\partial x} + \hat{\jmath}\frac{\partial v}{\partial y} + \hat{k}\frac{\partial v}{\partial z}\right)$$

$$W_{12} = \int_1^2 F.dr$$

$$= \int_1^2 -\nabla v \ dr$$

We know that

$$W_{12} = \int_1^2 \frac{dv}{dr}.dr$$

$$= -\int_1^2 dv$$

$$= -(V_2 - V_1)$$

$$W_{12} = V_1 - V_2 \qquad V \text{ – potential energy}$$

Which is the change in potential energy when the particle move in position 1 to 2.

(d) Conservation of energy:

Total energy is defines as a sum of kinetic energy and potential energy is constant for a conservative force.

$$W_{12} = V_1 - V_2$$

$$W_{12} = T_2 - T_1$$

$$T_2 - T_1 = V_1 - V_2$$

$$T_1 + V_1 = T_2 + V_2$$

Sum of potential energy and kinetic energy of a particle in conservative force field is constant.

2.2 Mechanics of a system of particles :

(more than one particle is represented by a coordinate)

Let consider a system have a n particles, the mass of the particle is m_1, m_2,………….,

m_i - mass of i^{th} particle

the position of the masse are represent r_1,, r_2, r_3, …………..,r_i

r_i - position of the i^{th} particle

Centre of mass is defined as the whole mass of the body or system assumes to be concentrated.

Consider two particle system at the position r_1 and r_2

Centre of mass represent the centre of point between the two masses.

Position of the centre of mass R = $\frac{m_1 r_1 + m_2 r_2}{m_1 + m_2}$

For an system of a particle,

Position of the centre of mass R = $\frac{m_1 r_1 + m_2 r_2 + m_3 r_3 + \ldots\ldots\ldots + m_n r_n}{m_1 + m_2 + m_3 + \cdots\ldots\ldots + m_n}$

$$R = \frac{\sum_{i=1}^{n} m_i r_i}{\sum_{i=1}^{n} m_i}$$

$$R = \frac{\sum_{i=1}^{n} m_i r_i}{M}$$

M - total mass of the system

$$M = m_1 + m_2 + m_3 + \cdots \ldots \ldots \ldots \ldots \ldots . + m_n$$

Newtons laws are valid for a system of particle.

The force acting on the i^{th} particle and the internal force are also acting on a i^{th} particle by another particles.

$$F_i = F_{ie} + \sum_{j=1}^{n} F_{ij}$$

From Newton's third law ($F_{12} = - F_{21}$)

F_{ij} - force acting on a i^{th} particle by the j particle.

From newton's second law

$$F_i = \frac{dp_i}{dt} = \dot{p_\iota}$$

$$= \frac{d \sum m_i v_i}{dt} \qquad (v_i = \dot{r_i})$$

$$= \frac{d \sum m_i \dot{r_\iota}}{dt}$$

$$= m_i \sum \frac{d^2 r_i}{dt^2}$$

Conservation of linear momentum:

The centre of the mass R of the system by

$$R = \frac{\sum_{i=1}^{n} m_i r_i}{M}$$

R is the position of the centre mass

$$R M = \sum_{i=1}^{n} m_i r_i$$

Differentiate above equation with respect to t

$$M\frac{dR}{dt} = m_1\frac{dr_1}{dt} + m_2\frac{dr_2}{dt} + m_3\frac{dr_3}{dt} + \ldots\ldots\ldots\ldots + m_n\frac{dr_n}{dt}$$

$$= m_1 v_1 + m_2 v_2 + m_3 v_3 + \ldots\ldots\ldots\ldots\ldots\ldots + m_n v_n$$

$$M\frac{dR}{dt} = \sum m_i v_i = p$$

From this equation total linear momentum is represent as the product of total mass of the system and total velocity of the system with the centre of mass

$$F = \frac{dp}{dt} = \frac{d(MV)}{dt}$$

$$= M\frac{dV}{dt} \qquad \left(V = \frac{dR}{dt} \right)$$

$$= M\frac{d^2R}{dt^2}$$

Total external force acting on a system of the particle is zero, the linear momentum of the system is conserved.

Conservation of angular momentum:

Consider the i^{th} particle mass m_i , and velocity v_i

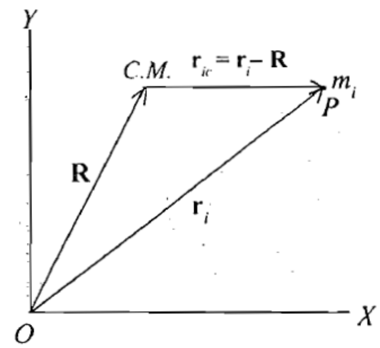

Fig.1.3

From above figure

$$r_{ic} = r_i - R$$

The angular momentum is written as

$$L = r \times p$$

For i^{th} particle

$$L_i = r_i \times p_i$$

$i = 1;\quad L_1 = r_1 \times p_1$

$i = 2;\quad L_2 = r_2 \times p_2$

$i = 3;\quad L_3 = r_3 \times p_3$

Differentiate it,

$$\frac{dL_i}{dt} = \frac{d(r_i \times \mathrm{P_i})}{dt}$$

$$\frac{dL_i}{dt} = \frac{d\,r_i}{dt} \times \mathrm{P_i} + r_i \times \frac{d\mathrm{P_i}}{dt}$$

$$(\ \mathrm{F_i} = \frac{d\mathrm{P_i}}{dt})$$

$$= v_i \times \mathrm{m}\, v_i + r_i \times \mathrm{F_i}$$

$$\frac{dL_i}{dt} = r_i \times \mathrm{Fi}$$

For the system of the particle

$$\frac{dL_i}{dt} = \sum_{i=1}^{n}(\ r_i \times F_i)$$

We know that

$$\mathrm{F_{ij}} = \mathrm{F_i} + \sum_{j=1}^{n} F_{ij}$$

$$\frac{dL}{dt} = \sum_{i=1}^{n}(\ r_i \times (F_i + \sum_{j=1}^{n} F_{ij}\ ))$$

$$\frac{dL}{dt} = \sum_{i=1}^{n}(\ r_i \times F_i) + \sum_{i=1}^{n} \sum_{j=1}^{n} (\ r_i \times F_{ij}\ )$$

from newton's third law $F_{ij} = -F_{ji}$

$$= (\ r_i \times F_i\ ) + (\ r_j \times F_{ji})$$

$$= r_i \times F_i - r_j \times F_{ij}$$

$$= (\ r_i - r_j\ ) \times F_{ij} = 0\ \}$$

$$\frac{dL}{dt} = \sum_{i=1}^{n}(\ r_i \times F_i)$$

$$= \tau$$

The external torque acting on a system is defined the rate of change of angular momentum.

$$\text{If} \quad \frac{dL}{dt} = 0;$$

$$\tau = \text{constant}$$

In absence of external torque the angular momentum is zero.

Conservation of energy:

(a) Kinetic energy:

Total kinetic energy of the system of particle is

$$= \sum_i \frac{1}{2} m_i v_i^2$$

$$= \sum_i \frac{1}{2} m_i ((\vec{v_i} \cdot \vec{v_i}))$$

We know that $v_i = v + v_{ic}$

$$\sum_i m_i \; r_{ic} = 0$$

$$= \sum_i \frac{1}{2} m_i (\mathrm{v} + v_{ic}) \; (\mathrm{v} + v_{ic})$$

$$= \frac{1}{2} \sum m \, (\mathrm{v} \, . \, \mathrm{v}) + \frac{1}{2} \sum m \, (\mathrm{v} \, . \, v_{ic}) + \frac{1}{2} \sum m \, (v_{ic} \, .\mathrm{v}) + \frac{1}{2} \sum m \, (v_{ic} \, . \, v_{ic})$$

$$= \sum_i \frac{1}{2} m_i v^2 + \sum_i \frac{1}{2} m_i v_{ic}^2$$

Kinetic energy $= \frac{1}{2} MV^2 + \sum_i \frac{1}{2} m_i v_{ic}^2$

The kinetic energy of the system of the particle is equal to kinetic energy of the system concentrated at the centre of mass and kinetic energy of the system at the centre of mass.

(b) Potential energy:

$$W_{12} = \sum_i \int_1^2 F_i^{\,e} \cdot dr_i + \int_1^2 \sum_i \sum_j F_{ij} \;\, dr_i \qquad (\mathrm{i} \neq j) \quad \text{-----------------(1)}$$

We know that $F = - \nabla v$

$$F_i^e = - \; \nabla_i \, v_i$$

$$F_{ij} = - \; \nabla_i \, v_{ij} \qquad \text{---------------(2)}$$

Take a first term,

$$\sum_i \int_1^2 F_i^e = \sum_i \int_1^2 - \; \nabla_i \, v_i \, dr_i$$

$$= \sum_i \int_1^2 - \; \frac{dv_i}{dr_i} dr_i$$

$$= \int_1^2 - \; dv_i$$

$$= - \sum_i [v_i]^2{}_1$$

$$= - \; [\; v_2 - \; v_1 \,]$$

$$= \; v_1 - \; v_2 \qquad \text{------------------(3)}$$

Take a second term

$$\int_1^2 \sum_i \sum_j F_{ij}\ dr_i = \frac{1}{2}\sum_i \sum_j \int_1^2 F_{ij}\ dr_i + F_{ji}\ dr_j$$

$$= \frac{1}{2}\sum_i \sum_j \int_1^2 F_{ij}\ (dr_i - dr_j)$$

$$dr_i - dr_j = dr_{ij}$$

$$= \frac{1}{2}\sum_i \sum_j \int_1^2 F_{ij}\ dr_{ij}$$

Substitute equation (2) in above equation

$$= \frac{1}{2}\sum_i \sum_j \int_1^2 (-\nabla_i v_{ij})\, dr_{ij}$$

$$= \frac{1}{2}\sum_i \sum_j \int_1^2 \left(-\frac{dv_{ij}}{dr_{ij}}\right) dr_{ij}$$

$$= \frac{1}{2}\sum_i \sum_j \int_1^2 -dv_{ij}$$

$$= -\frac{1}{2}\sum_i \sum_j \int_1^2 [v_{ij}]^2{}_1 \qquad \text{-------------------(4)}$$

Equation (4) and (3) are substitute in equation (1)

$$W_{12} = -\sum_i [v_i]^2{}_1 + \frac{1}{2}\sum_i \sum_j \int_1^2 [v_{ij}]^2{}_1$$

$$W_{12} = v_1 - v_2$$

Therefore, the total potential energy of the system is

$$V = \sum_i [v_i]^2{}_1 + \frac{1}{2}\sum_i \sum_j [v_{ij}]^2{}_1 \qquad \text{----------(5)}$$

Comparing the equation,

$$v_1 - v_2 = T_2 - T_1$$

$$T_1 + V_1 = T_2 + V_2$$

$$T + V = \text{constant}$$

Therefore, the total energy of the system is conserved.

The conservation of energy is defined as the total energy of the system is constant.

2.3 Constraints :

Often the motion of a particle of system of particle is restricted by one or more conditions. The limitation on the motion of a system is called constraints and the motion is said to be constraint motion.

2.4 Holonomic constraints:

Constraint limit the motion of a system and the number of independent coordinates, Needed to describe the motion, is reduced.

If r is the position vector of the particle at any angular coodinates θ relarive to the centre of the circle, then,

$$|r| = a$$

$$r - a = 0$$

The above equation express the constraint for a particle in circular motion.

Suppose the constraints are present in the system of N particles. If the constraints are expressed in the form of equations of the form

$$f(r_1, r_2, r_3, \ldots.., t) = 0$$

then they are called holonomic constraints.

Let there be m number of such equations to describe the constraints in the N particle system. Now, we use these equation to eliminate m of the 3N co-ordinates and we need only n independent coordinates to describe the motion, given by

$$n = 3N - m$$

2.5 non holonomic constraints:

In a non holonomic system, all the coordinates cannot vary independently and hence the number of degrees of freedom of the system is less than the minimum number of coordinates needed to specify the configuration of the system.

Example:

(i) The motion of the particle, placed on the surface of a radius a, will be described by

$$|r| \geq a$$

$$r - a \geq 0$$

In a gravitational field, where r is the position vector of the particle relative to the centre of the sphere, the particle will first slide down the surface and then fall off.

(ii) The gas molecules in a container are constrained to move inside it and related constraint is another example of non holonomic constraints.

If the gas container is in spherical shape with radius a and r is the position vector of a molecule, then the condition of constraint for the motion of a molecule can be expressed as

$$|r| \leq a$$

$$r - a \leq 0$$

Example:

In the following cases, discuss whether the constraint is holonomic of nonholonomic. Specify the constraint force also.

(i) Motion of the body on a inclined plane under gravity.

(ii) A bead on a circular wire.

(iii) A particle moving on an ellipsoid under the influence of gravity

(iv) A pendulum with variable length.

Solution:

(i) When a body moves on an inclined plane, it is constrained to move on the inclined plane surface. Hence the constraint is holonomic. The force of constraint is the reaction of the plane, acting normal to the inclined surface.

(ii) The constraint is that the bead remains at a constant distance a, the radius of the circular wire and can be expressed as $r = a$. hence the constraint is holonomic. The force of constraint is the reaction of the wire, acting on the bead.

(iii) The constraint is non holonomic, because the particle after reaching a certain point will leave the ellipsoid. Force of constraint is the reaction force of the ellipsoid surface on the particle.

(iv) The constraint is holonomic. Because the constraint equation is $|r| = L(t)$. the constraint force is the tension in the string

2.6 Generalized coordinates:

Set of independent coordinates sufficient in number to describe the state of configuration of a dynamical system.

These coordinates are denoted as,

$$q_1, q_2, q_3, \ldots\ldots\ldots\ldots\ldots, q_n$$

The generalized coordinates for a system of N particles, constrained by m equations, are

$$n = 3N - m$$

It is not necessary that these coordinates should be rectangular, spherical or cylindrical.

For a system of N particles, if r_1, r_2, r_3 are the cartesian coordinates of the i^{th} particle, then these coordinates in terms of the generalized coordinates q_k can be expressed as,

$$r_1 = r_1\,(q_1, q_2, q_3, \ldots\ldots\ldots\ldots\ldots, q_n, t)$$
$$r_2 = r_2\,(q_1, q_2, q_3, \ldots\ldots\ldots\ldots\ldots, q_n, t)$$
$$r_3 = r_3\,(q_1, q_2, q_3, \ldots\ldots\ldots\ldots\ldots, q_n, t)$$

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$$r_n = r_n\,(q_1, q_2, q_3, \ldots\ldots\ldots\ldots\ldots, q_n, t)$$

some example of the mechanical system with the corresponding generalized coordinates,

(1) Rotating mass (m) at the end of a string in circle of radius (r): one generalized coordinate $q_1 = \theta$. Where θ is the angle between rotating string and fixed particular radius.

(2) Simple pendulum: There is one generalized coordinates $q_1 = \theta$. Where θ is the angle which the thread of the pendulum makes with the vertical line through the point of suspension.

(3) Fly wheel: Only one generalized coordinates $q_1 = \theta$ where, θ is the angle between a particular radius of the fly wheel and fixed line perpendicular to its axis.

(4) Beads of an abacus: One generalized coordinates $q_1 = x$ along the horizontal wire.

(5) Particle moving on the surface of a sphere: $q_1 = \theta$, $q_2 = \varphi$ are two generalized coordinates.

(6) Particle moving on the surface of a sphere: $q_1 = r$, $q_2 = \theta$ are two generalized coordinates, where r is the radius drawn from the vertex as origin to the position of the particle and θ is the angle of the radius vector with a fixed slant edge of the cone.

2.7 Configuration space:

The configuration of the system of N particles, moving freely in space, may be represented by the position of a single point in 3N dimensional space, which is called configuration space of the system.

The configuration space for a system of one freely moving particle is 3 - dimensional and for a system of two freely moving particles, it is six dimensional.

In later case, the configuration of the system of the two particles can be represented by the position of a single point with six coordinates in six dimensional space. This system has six degrees of freedom and its configuration space is six dimensional.

The degrees of freedom of a dynamical system is defined as the minimum number of independent coordinates required to specify the system compatible with the constraints.

If there are n independent variables, say $q_1, q_2, q_3, \ldots\ldots, q_n$ and n constants $c_1, c_2, c_3, \ldots\ldots, c_n$ such that

$$\sum_{i=1}^{n} c_i \, dq_i = 0$$

At any position of the system, then we must have

$$c_1 = c_2 = \ldots\ldots = c_n = 0$$

2.8 Principle of virtual work:

Consider the i^{th} particle. Position vector of i^{th} particle is r_i and the generalized coordinate of i^{th} particle is

$$r_i = r_i\,(q_1, q_2, q_3, \ldots\ldots, q_n, t)$$

In virtual displacement does not involve the time. The virtual displacement of the i^{th} particle is represented by

$$\delta r_i = \partial r_i\,(q_1, q_2, q_3, \ldots\ldots, q_n, t)$$

Force acting on the i^{th} particle is

Force = external force + force of constraints

$$F_i = F_i^a + F_i \qquad \text{---------------(1)}$$

F_i^a - applied force

If the system is in equilibrium then the total force acting on a system is zero.

i.e., $F_i = 0$

the sum of virtual work is

$$\delta \mathrm{W} = \sum_i F_i \ \delta \mathrm{r_i} = 0$$

Substitute equation(1) in above equation, we get

$$F_i = \sum_i \ (F_i^a + F_i) \ \delta \mathrm{r_i} = 0$$

$$\mathrm{W} = \sum_i F_i^a \ \delta \mathrm{r_i} + \sum_i F_i \ \delta \mathrm{r_i} = 0$$

$\sum_i F_i^a \ \delta \mathrm{r_i}$ - work done by the external force

$\sum_i F_i \ \delta \mathrm{r_i}$ - work done by force of constraint

Work done by the force of constraint is zero and a virtual displacement is written as

$$\delta \mathrm{W} = \sum_i F_i^a \ \delta \mathrm{r_i} = 0$$

The principle of virtual work states that the total work done is zero for a system in equilibrium. Principle of virtual work is deals with its only with states of a system of particles.

UNIT II: LAGRANGIAN FORMULATION

D'Alembert's principle – Lagrangian equations of motion for conservative systems – applications: (i) simple pendulum (ii)Atwood's machine – Lagrange's equations in presence of non- –Lagrangian for a charged particle moving in an Electromagnetic field.

2.1 D'Alembert's principle:

It states that any system is in equilibrium and a force is equal to the sum of actual force and the reverse effective force.

D'Alembert's tried to introduce the motion of the system of particle.

Let consider the motion of a system of n particle. F_i force acting on the i^{th} particle. According to Newton's second law,

$$F_i = \frac{dp_i}{dt} = \dot{p}_i \quad \text{(dynamic)}$$

$$F_i + (-\dot{p}_i) = 0 \quad \text{(statistic)}$$

$$F_i - \dot{p}_i = 0$$

$-\dot{p}_i$ - reverse effective force

The above equation represent the i^{th} particle in the system in equilibrium and the force is equal to the actual force (F_i) and the reverse effective force ($-\dot{p}_i$).

For a virtual displacement δr_i

$$\sum_{i=1}^{n}(F_i - \dot{p}_i)\,\delta r_i = 0$$

$$(F_i = F_i^a + f_i)$$

$$\sum_{i=1}^{n}((F_i^a + f_i) - \dot{p}_i)\,\delta r_i = 0$$

$$\sum_{i=1}^{n}(F_i^a - \dot{p}_i)\,\delta r_i + \sum_{i=1}^{n} f_i\,\delta r_i = 0$$

$\sum_{i=1}^{n} f_i\,\delta r_i$ - total work done by the force of constraint is zero.

$$\sum_{i=1}^{n}(F_i^a - \dot{p}_i)\,\delta r_i = 0$$

The above equation represents the D'Alembert's principle.

Since the force of constraints do not appear in the equation and hence now we can drop the superscript. Therefore, the D'Alembert's principle may be written as,

$$\sum_{i=1}^{n}(F_i - \dot{p}_i)\,\delta r_i = 0$$

2.2 Lagrangian equations of motion

Let consider a system of N particles $r_1, r_2, r_3, \ldots\ldots., r_n$ having m constraints.

The number of independent coordinate is

$$n = 3N - m$$

The generalized coordinate are represent as

$$q_1, q_2, q_3, \ldots\ldots\ldots\ldots\ldots, q_n$$

Therefore the transformation equation representing the position of i^{th} particle ,

$$r_i = r_i\,(q_1, q_2, q_3, \ldots\ldots\ldots\ldots\ldots, q_n, t) \quad \text{------------(1)}$$

Virtual displacement δr_i is given by

$$\delta r_i = \frac{\partial r_i}{\partial q_1}\,\delta q_1 + \frac{\partial r_i}{\partial q_2}\,\delta q_2 + \ldots\ldots\ldots. + \frac{\partial r_i}{\partial q_n}\,\delta q_n + \frac{\partial r_i}{\partial t}\,\delta t$$

It is time independent

$$\delta r_i = \frac{\partial r_i}{\partial q_1}\,\delta q_1 + \frac{\partial r_i}{\partial q_2}\,\delta q_2 + \ldots\ldots\ldots. + \frac{\partial r_i}{\partial q_n}\,\delta q_n$$

$$\delta r_i = \sum_{i=1}^{n}\frac{\partial r_i}{\partial q_k}\,\delta q_k \qquad \text{----------------(2)}$$

Differentiate equation (1) with respect to time

$$\frac{dr_i}{dt} = \frac{\partial r_i}{\partial q_1}\frac{dq_1}{dt} + \frac{\partial r_i}{\partial q_2}\frac{dq_2}{dt} + \ldots\ldots\ldots\ldots\ldots + \frac{\partial r_i}{\partial q_n}\frac{dq_n}{dt} + \frac{\partial r_i}{\partial t}$$

$$\dot{r_i} = \frac{\partial r_i}{\partial q_1}\,\dot{q}_1 + \frac{\partial r_i}{\partial q_2}\,\dot{q}_2 + \ldots\ldots\ldots\ldots\ldots\ldots + \frac{\partial r_i}{\partial q_n}\,\dot{q}_n + \frac{\partial r_i}{\partial t}$$

$$\dot{r_i} = v_i = \sum_{k=1}^{n}\frac{\partial r_i}{\partial q_k}\,\dot{q}_k + \frac{\partial r_i}{\partial t} \qquad \text{------------(3)}$$

Differentiate with respect to $\dot{q}_k$

$$\frac{\partial r_i}{\partial \dot{q}_k} = \sum_{k=1}^{n}\frac{\partial r_i}{\partial \dot{q}_k}\,\dot{q}_k \quad \text{--------------(4)}$$

According to D'Alembert's principle,

$$\sum_{i=1}^{n}\,(F_i - \dot{p_i})\,\delta r_i = 0$$

$$\sum_{i=1}^{n} F_i\,\delta r_i - \sum_{i=1}^{n}\dot{p_i}\,\delta r_i = 0 \quad \text{------------- (5)}$$

Take first term of equation (5)

$$\sum_{i=1}^{n} F_i \quad \delta r_i \quad = \quad \sum_{i=1}^{n} F_i \quad \left(\sum_{k=1}^{n} \frac{\partial r_i}{\partial q_k} \ \delta q_k \right)$$

$$= \quad \sum_{k=1}^{n} \left(\ \sum_{i=1}^{n} \quad F_i \frac{\partial r_i}{\partial q_k} \right) \ \delta q_k$$

$$\sum_{i=1}^{n} F_i \quad \delta r_i \quad = \quad \sum_{k=1}^{n} G_k \ \ \delta q_k \qquad \text{-------------- (6)}$$

Where $G_k = \ \sum_{i=1}^{n} \quad F_i \frac{\partial r_i}{\partial q_k}$ -------------- (7)

G_k - generalized force

Take second term of equation (5)

$$\sum_{i=1}^{n} \ \dot{p}_i \ \ \delta r_i \quad = \ \sum_{i=1}^{n} m_i \ \ddot{r}_i \ \ \delta r_i$$

$$= \ \sum_{i=1}^{n} m_i \ \frac{\partial \dot{r}_i}{\partial t} \ \ \delta r_i$$

$$= \ \sum_{i=1}^{n} m_i \ \ddot{r}_i \ \sum_{k=1}^{n} \frac{\partial r_i}{\partial q_k} \ \ \delta q_k$$

$$= \ \sum_{k=1}^{n} \left(\ \sum_{i=1}^{n} \frac{\partial r_i}{\partial q_k} \ m_i \ \ddot{r}_i \ \right) \ \delta q_k \ \text{--------- (8)}$$

$$\left\{ \quad \frac{d}{dt}\left(m_i \ \dot{r}_i \ \frac{\partial r_i}{\partial q_k} \right) \ = \ m_i \ \ddot{r}_i \ \frac{\partial r_i}{\partial q_k} + \ m_i \ \dot{r}_i \ \frac{d}{dt}\left(\frac{\partial r_i}{\partial q_k}\right) \right.$$

$$\left. m_i \ \ddot{r}_i \ \frac{\partial r_i}{\partial q_k} \ = \ \frac{d}{dt}\left(m_i \ \dot{r}_i \ \frac{\partial r_i}{\partial q_k} \right) \ - \ m_i \ \dot{r}_i \ \frac{d}{dt}\left(\frac{\partial r_i}{\partial q_k}\right) \ \right\}$$

Above equation is substitute in equation(8), we get

$$\sum_{i=1}^{n} \ \dot{p}_i \ \ \delta r_i \ = \sum_{k=1}^{n}\left(\ \sum_{i=1}^{n} \frac{d}{dt}\left(m_i \ \dot{r}_i \ \frac{\partial r_i}{\partial q_k} \right) - \ \sum_{i=1}^{n} m_i \ \dot{r}_i \ \frac{d}{dt}\left(\frac{\partial r_i}{\partial q_k}\right) \right) \ \delta q_k \ \text{---------(9)}$$

Substitute equation (4) in (9) we get

$$\sum_{i=1}^{n} \ \dot{p}_i \ \ \delta r_i \ = \sum_{k=1}^{n}\left(\ \sum_{i=1}^{n} c - \ \sum_{i=1}^{n} m_i \ \dot{r}_i \ \frac{d}{dt}\left(\frac{\partial \dot{r}_i}{\partial \dot{q}_k}\right) \right) \ \delta q_k$$

$$= \sum_{k=1}^{n}\left(\ \sum_{i=1}^{n} \frac{d}{dt}\left(m_i \ \dot{r}_i \ \frac{\partial \dot{r}_i}{\partial \dot{q}_k} \right) - \ \sum_{i=1}^{n} m_i \ \dot{r}_i \ \frac{d}{d\dot{q}_k}\left(\frac{\partial \dot{r}_i}{\partial t}\right) \right) \ \delta q_k$$

$$= \sum_{k=1}^{n}\left(\ \sum_{i=1}^{n} \frac{d}{dt}\left(m_i \ \dot{r}_i \ \frac{\partial \dot{r}_i}{\partial \dot{q}_k} \right) - \ \sum_{i=1}^{n} m_i \ \dot{r}_i \ \frac{d\dot{r}_i}{d\dot{q}_k} \ \right) \ \delta q_k$$

$$= \sum_{k=1}^{n}\left(\sum_{i=1}^{n}\frac{d}{dt}\left(\frac{\partial}{\partial \dot{q_k}} \left(\frac{1}{2}\, m_i\, \dot{r}^2{}_i\right) \right) - \sum_{i=1}^{n} \frac{d}{d\dot{q_k}} \left(\frac{1}{2}\, m_i\, \dot{r}^2{}_i\right) \right) \delta q_k$$

$\frac{1}{2}\, m_i\, \dot{r}^2{}_i =$ T - is kinetic energy of the system,

$$\sum_{i=1}^{n} \dot{p_\iota}\ \delta r_i = \sum_{k=1}^{n}\left(\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q_k}} \right) - \frac{dT}{d\dot{q_k}}\right) \delta q_k \quad \text{----------------(10)}$$

Substitute equation(10) and (6) in (5), we get

$$\sum_{k=1}^{n} G_k\ \delta q_k - \sum_{k=1}^{n}\left(\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q_k}} \right) - \frac{dT}{d\dot{q_k}}\right) \delta q_k = 0$$

$$\sum_{k=1}^{n} \left\{ \left(\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q_k}} \right) - \frac{dT}{dq_k}\right) - G_k \right\} \delta q_k = 0 \quad \text{------------ (11)}$$

Since δq_k is independent of each other, their coefficient must be zero.

$$\left(\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q_k}} \right) - \frac{dT}{dq_k}\right) - G_k = 0$$

$$\boldsymbol{\left(\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q_k}} \right) - \frac{dT}{dq_k}\right) = G_k} \quad \text{----------------- (12)}$$

Above equation represent **the general form of Lagrangian's equation**.

2.3 Lagrangian equations of motion for conservative systems:

Let consider a i[th] particle, the force acting on the i[th] particle

$$F_i = -\nabla_i v = \frac{\partial v}{\partial r_i} \quad \text{----------------(13)}$$

$$G_k = \sum_{i=1}^{n} F_i \frac{\partial r_i}{\partial q_k}$$

$$= \sum_{i=1}^{n} (-\nabla_i v) \frac{\partial r_i}{\partial q_k}$$

$$= \sum_{i=1}^{n} -\left(\frac{\partial v}{\partial r_i}\right) \frac{\partial r_i}{\partial q_k}$$

$$= -\frac{\partial v}{\partial q_k} \quad \text{--------------------(14)}$$

Substitute equation (14) in (12), we get

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q_k}} \right) - \frac{dT}{dq_k} = -\frac{\partial v}{\partial q_k}$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q_k}}\right) - \frac{dT}{dq_k} + \frac{\partial v}{\partial q_k} = 0$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q_k}}\right) - \frac{d\,(T-V)}{dq_k} = 0$$

V – potential energy

Since the potential is the function of generalized coordinate q_k and not dependent of generalized velocity.

$$\text{i.e.,} \quad \frac{\partial v}{\partial \dot{q_k}} = 0$$

$$\frac{d}{dt}\left(\frac{\partial\,(T-v)}{\partial \dot{q_k}}\right) - \frac{d\,(T-V)}{dq_k} = 0$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q_k}}\right) - \frac{dL}{dq_k} = 0 \quad \text{----------------(15)}$$

The above equation represent the Lagrangian equation for a conservative system.

L = T + V

The Lagrangian is the function of L = L ($\dot{q}$, q, t)

(velocity, position, time)

L ($\dot{q}$, q, t) = T ($\dot{q}$, q, t) + V(q)

2.4 Lagrange's equations in presence of non- conservative forces:

Let us consider the system acted upon by conservative force F_i and non – conservative force f_i the component of generalized force is represent by

$$G_k = \sum_{i=1}^{n} F \cdot \frac{\partial r_i}{\partial q_k}$$

$$F = F_i + f_i$$

$$G_k = \sum_{i=1}^{n} (F_i + f_i) \cdot \frac{\partial r_i}{\partial q_k}$$

$$G_k = \sum_{i=1}^{n} F_i \frac{\partial r_i}{\partial q_k} + \sum_{i=1}^{n} f_i \frac{\partial r_i}{\partial q_k}$$

For conservative system,

$$F_i = -\nabla_i v = \frac{\partial v}{\partial r_i}$$

$$G_k = \sum_{i=1}^{n} F_i \frac{\partial r_i}{\partial q_k}$$

$$= \sum_{i=1}^{n} (-\nabla_i v) \frac{\partial r_i}{\partial q_k}$$

$$= \sum_{i=1}^{n} -\left(\frac{\partial v}{\partial r_i}\right) \frac{\partial r_i}{\partial q_k}$$

$$= -\frac{\partial v}{\partial q_k}$$

$$G_k = -\frac{\partial v}{\partial q_k} + G_k{}'$$

$$G_k' = \sum_{i=1}^{n} f_i \frac{\partial r_i}{\partial q_k}$$

The generalized form of Lagrangian equation is

$$\left(\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) - \frac{dT}{dq_k}\right) = G_k$$

$$\left(\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) - \frac{dT}{dq_k}\right) = -\frac{\partial v}{\partial q_k} + G_k{}'$$

$$\frac{\partial v}{\partial q_k} = 0$$

$$\frac{d}{dt}\left(\frac{\partial (T-v)}{\partial \dot{q}_k}\right) - \frac{d(T-V)}{dq_k} = G_k{}'$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{dL}{dq_k} = G_k{}'$$

The above equation represents the Lagrangian equation in presence of non – conservative force.

Applications:

2.5 (i) Simple pendulum:

Let θ be the angular displacement of the simple pendulum from the equilibrium position. If l be the effective length of the pendulum and m be the mass of the bob, then the displacementalong arc OA = s.

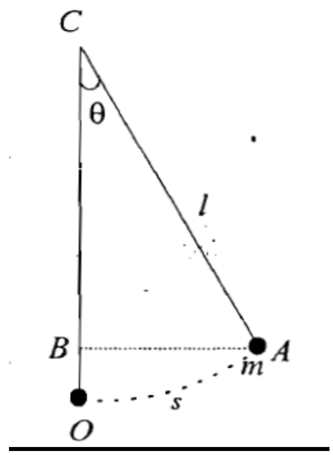

Fig. 2.4

Lagrangian equation is given by,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{dL}{d\theta} = 0 \quad \text{---------------(1)}$$

$$L = T + V; \qquad T = \frac{1}{2} m v^2$$

Where $$\theta = \frac{arc\ length}{original\ length}$$

$$= \frac{r}{l}$$

The velocity is represent as v = $\frac{dr}{dt} = \frac{d(l\,\theta)}{dt}$

$$= l\,\frac{d\theta}{dt}$$

$$= l\,\dot{\theta}$$

$$T = \frac{1}{2} m v^2 \quad = \frac{1}{2} m\,(l\,\dot{\theta})^2$$

$$= \frac{1}{2} m\, l^2\, \dot{\theta}^2 \quad \text{--------------(2)}$$

Potential v = mgh

$$OB = OC - CB$$

$$h = l - l\cos\theta$$

$$h = l\,(1 - \cos\theta)$$

potential energy is given by

$$V = -mg\,l\,(1 - \cos\theta) \qquad \text{----------------(3)}$$

We know that

$$L = T + V$$

$$L = \frac{1}{2} m\,l^2\,\dot{\theta}^2 - mg\,l\,(1 - \cos\theta) \qquad \text{------------(4)}$$

Substitute equation(4) in (1)

$$\frac{d}{dt}\left(\frac{\partial(\frac{1}{2} m\,l^2\,\dot{\theta}^2 - mg\,l\,(1 - \cos\theta\,)}{\partial\dot{\theta}}\right) - \frac{d(\frac{1}{2} m\,l^2\,\dot{\theta}^2 - mg\,l\,(1 - \cos\theta\,)}{d\theta} = 0$$

$$\frac{d}{dt}\left(\frac{\partial(\frac{1}{2} m\,l^2\,\dot{\theta}^2 - mg\,l\,(1 - \cos\theta\,)}{\partial\dot{\theta}}\right) - \frac{d(\frac{1}{2} m\,l^2\,\dot{\theta}^2 - mg\,l\, - mg\cos\theta\,)}{d\theta} = 0$$

$$\frac{d(m\,l^2\,\dot{\theta})}{dt} + mg\,l\sin\theta = 0$$

$$m\,l^2\,\ddot{\theta} + mg\,l\sin\theta = 0$$

Above equation is divided by $m\,l$

$$l\,\ddot{\theta} + g\sin\theta = 0$$

$$\mathrm{Sin}\theta \cong \theta$$

$$l\,\ddot{\theta} + g\,\theta = 0$$

Above equation is divided by l

$$\ddot{\theta} + \frac{g}{l}\theta = 0$$

The above equation represent the equation of motion of a simple pendulum.

$$\ddot{\theta} + \omega^2 \theta = 0$$

Where $\omega^2 = \frac{g}{l}$

$$\omega = \sqrt{\frac{g}{l}}$$

$$T = \frac{2\pi}{\omega}$$

$$T = \frac{2\pi}{\sqrt{\frac{g}{l}}} = 2\pi\sqrt{\frac{l}{g}}$$

This is represent a simple harmonic motion of period.

(ii) Atwood's machine:

The atwood's machine is the example of a conservative system with holonomic constraints. The pulley is small, massless and frictionless.

Let the two masses m_1 and m_2 which are connected by an inextensible string of length l. Suppose x be the variable vertical distance from the pully to the mass m_1 then mass m2 will be at a distance l - x from the pulley.

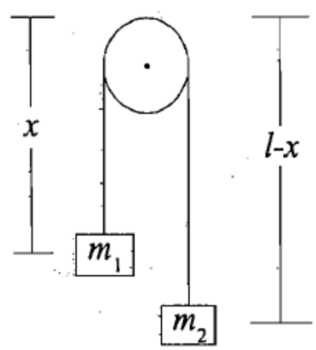

Fig. 2.5

The velocities of two masses

$$V_1 = \frac{dx}{dt} = \dot{x}$$

There is only one independent variable

$$V_2 = \frac{d(l - x)}{dt} = \frac{dl}{dt} - \frac{dx}{dt} = 0 - \dot{x} = -\dot{x}$$

Kinetic energy of the system is

$$T = \frac{1}{2} m_1 \dot{x}^2 + \frac{1}{2} m_2 \dot{x}^2$$

$$= \frac{1}{2} \dot{x}^2 \ (m_1 + m_2)$$

Potential energy of the system is

$$V = - m_1 g x - m_2 g (l - x)$$

Thus the lagrangian is

$$L = T - V$$

$$= \frac{1}{2} m_1 \dot{x}^2 + \frac{1}{2} m_2 \dot{x}^2 + m_1 g x + m_2 g (l - x)$$

Lagrangian equation is given by,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{dL}{dx} = 0$$

$$\frac{d}{dt}\left(\frac{\partial(\frac{1}{2} m1 \ \dot{x}^2 + \frac{1}{2} m2 \ \dot{x}^2 + m1gx + m2g (l - x))}{\partial \dot{\theta}}\right) - \frac{d(\frac{1}{2} m1 \ \dot{x}^2 + \frac{1}{2} m2 \ \dot{x}^2 + m1gx + m2g (l - x))}{d\theta} = 0$$

$$\frac{d}{dt}\left(\dot{x} \ (m_1 + m_2)\right) - (m_1 \, g - m_2 \, g) = 0$$

$$\frac{d}{dt}\left(\dot{x} \ (m_1 + m_2)\right) - g (m_1 - m_2) = 0$$

$$\ddot{x} (m_1 + m_2) - g (m_1 - m_2) = 0$$

$$\ddot{x} (m_1 + m_2) = g (m_1 - m_2)$$

$$\ddot{x} = \frac{(m_1 - m_2)}{(m_1 + m_2)} \ g$$

If $m_1 > m_2$ the mass m_1 decends with constant acceleration.

If $m_1 < m_2$ the mass m_1 ascends with constant acceleration.

(iii) Lagrangian for a charged particle moving in an electromagnetic field:

The force acting on a charge q, moving with velocity v in an electric field E and magnetic induction B is given by

$$F = q\ (\ E + (\ v \times\ B\ )) \quad \text{---------------- (1)}$$

Where $E = -\nabla\varphi - \frac{\partial A}{\partial t}$ ------------------ (2)

φ - electric scalar potential

A - magnetic vector potential

$$B = \nabla \times\ A \quad \text{----------------------(3)}$$

Substitute equation (2) and (3) in equation (1), we get

$$F = q\ (-\nabla\varphi - \frac{\partial A}{\partial t} + (\ v \times \nabla \times\ A\ )) \quad \text{--------- (4)}$$

Let us consider the x – component.

Since, $\nabla\varphi = \hat{\imath}\frac{\partial\varphi}{\partial x} + \hat{\jmath}\frac{\partial\varphi}{\partial y} + \hat{z}\frac{\partial\varphi}{\partial z}$

From equation (4) we take

$$(\ v \times \nabla \times\ A\ )_x = v_y\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) - v_z\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)$$

Add and subtract the term $v_x\frac{\partial A_x}{\partial x}$

$$(\ v \times \nabla \times\ A\ )_x = v_y\frac{\partial A_y}{\partial x} - v_y\frac{\partial A_x}{\partial y} - v_z\frac{\partial A_x}{\partial z} + v_z\frac{\partial A_z}{\partial x} + v_x\frac{\partial A_x}{\partial x} - v_x\frac{\partial A_x}{\partial x} \quad \text{----------(5)}$$

$$\frac{dA_x}{dy} = \frac{\partial A_x}{\partial x}\frac{dx}{dt} + \frac{\partial A_x}{\partial y}\frac{dy}{dt} + \frac{\partial A_x}{\partial z}\frac{dz}{dt} + \frac{\partial A_x}{\partial t}$$

$$\frac{dA_x}{dy} = \frac{\partial A_x}{\partial x}v_x + \frac{\partial A_x}{\partial y}v_y + \frac{\partial A_x}{\partial z}v_z + \frac{\partial A_x}{\partial t}$$

$$\frac{dA_x}{dy} - \frac{\partial A_x}{\partial t} = \frac{\partial A_x}{\partial x}v_x + \frac{\partial A_x}{\partial y}v_y + \frac{\partial A_x}{\partial z}v_z \quad \text{-------------(6)}$$

$$\frac{\partial(v.A)}{\partial x} = \frac{\partial(v_x A_x + v_y A_y + v_z A_z)}{\partial x}$$

$$= \frac{\partial A_x}{\partial x}v_x + \frac{\partial A_x}{\partial y}v_y + \frac{\partial A_x}{\partial z}v_z \quad \text{--------------(7)}$$

Substituting equation(6) and (7) in (5), we get

$$(v \times \nabla \times A)_x = \frac{\partial(v .A)}{\partial x} - \frac{dA_x}{dy} + \frac{\partial A_x}{\partial t} \quad \text{--------------(8)}$$

Hence from equation (4) , the x – component of the force F is

$$F_x = q (- \frac{\partial\varphi}{\partial x} - \frac{\partial A_x}{\partial t} + \frac{\partial(v .A)}{\partial x} - \frac{dA_x}{dy} + \frac{\partial A_x}{\partial t})$$

$$= q (- \frac{\partial(\varphi - v . A)}{\partial x} - \frac{\partial A_x}{\partial t}) \quad \text{--------------(9)}$$

Since $\frac{\partial(v . A)}{\partial v_x} = \frac{\partial(v_x A_x + v_y A_y + v_z A_z)}{\partial v_x} = A_x$

And scalar potential φ is independent of v_x, we have

$$- \frac{\partial A_x}{\partial t} = \frac{d}{dt}\frac{\partial}{\partial v_x}(\varphi - v . A)$$

$$F_x = q [\frac{\partial}{\partial x}(\varphi - v . A) + \frac{d}{dt}(\frac{\partial}{\partial v_x}(\varphi - v . A))] \quad \text{------------(10)}$$

we defined a generalized potential U, given by

$$U = q (\varphi - v . A) \quad \text{-----------------(11)}$$

Which is velocity dependent potential

$$F_x = - \frac{\partial U}{\partial x} + \frac{d}{dt}\frac{\partial U}{\partial v_x} \quad \text{---------------(12)}$$

The lagrangian equation is

$$\frac{d}{dt}\left(\frac{\partial T}{\partial v_x} \right) - \frac{dT}{dx} = F_x \quad \text{-------------(13)}$$

Substitute equation(12) in (13) we get

$$\frac{d}{dt}\left(\frac{\partial(T - U)}{\partial \dot{x}} \right) - \frac{d(T - U)}{dx} = 0$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}} \right) - \frac{dL}{dx} = 0$$

$$L = T - U$$

$$= T - q (\varphi - v . A)$$

$$= T - q \varphi - qv . A$$

The above equation is represent the charged particle moving in an electromagnetic field.

UNIT III: HAMILTONIAN FORMULATION

Phase space – generalized momentum and cyclic coordinates – Hamiltonian function and conservation of energy – Hamilton's canonical equations of motion – applications: (i) one dimensional simple harmonic oscillator (ii) motion of particle in a central force field.

3.1 Phase space:

In the Hamiltonian formulation, we observe from the equations of motion that the momentum coordinates p_k (k = 1, 2, 3, …………,n) and position coordinates q_k (k = 1, 2, 3, …………,n) play similar roles.

We can imagine a space in 2n dimension, in which a complete dynamical specification of a mechanical system is given by a point, having 2n coordinates ($p_1, p_2, \ldots p_n, q_1, q_2, \ldots q_n$).
Such a space is known as 2n – dimensional phase space.

If we know the state of a mechanical system at time t. i.e., we know all position and momentum coordinates, then this state will be represented by a specific point in the phase space. In other words, a point in phase space specifies the state of a mechanical system.

Symbolically, the representative point for the state of the system in the phase space can be written as

$$r = (q_1, q_2, \ldots q_n, p_1, p_2, \ldots p_n).$$

As the time advances, the changing state of the system may be described by a curve r(t) in the phase space. This is called phase path.

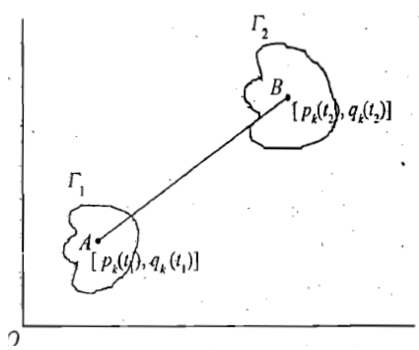

Fig. 3.1

3.2 Generalized momentum:

Consider a ith particle mass m, it moving with velocity v and it moving in along x – direction.

The kinetic energy of the particle is

$$T = \frac{1}{2} m \dot{x}^2$$

$$\frac{dT}{d\dot{x}} = m \dot{x}$$

We know that, the potential energy V is a function of x alone.

$$\frac{dV}{d\dot{x}} = 0$$

$$\frac{dT}{d\dot{x}} - \frac{dV}{d\dot{x}} = \frac{\partial(T-V)}{\partial \dot{x}}$$

$$= m \dot{x}$$

$$\frac{dT}{d\dot{x}} = \frac{dV}{d\dot{x}} = m \dot{x}$$

$$= p_x$$

p_x - generalized momentum

Therefore, a system describe set of the generalized coordinates q_k and generalizd velocities $\dot{q}_k$.

Then define the generalized momentum corresponding to the generalized coordinate q_k *as*

$$p_k = \frac{\partial L}{\partial \dot{q}_k}$$

It also called conjugate momentum and canonical momentum.

Lagrangian equation of motion for a conservative system is given by,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = 0$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) = \frac{dL}{\partial q_k}$$

$$\left(\frac{dp_k}{d\dot{q}_k}\right) = \frac{\partial L}{\partial q_k}$$

$$\dot{p}_k = \frac{\partial L}{\partial \dot{q}_k}$$

3.3 Cyclic or ignorable coordinates:

Coordinates that do not appear explicity in the lagrangian of a system is called cyclic coordinate of ignorable coordinate.

If q_k is the cyclic coordinate,

p_k does not appear the lagrangian of a system.

Thus, $\frac{\partial L}{\partial q_k} = 0$

q_k - cyclic coordinate

p_k - conjugate momentum

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = 0$$

$$\left(\frac{dL}{\partial q_k} = 0 \right)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) = 0$$

$$\frac{dp_k}{dt} = 0$$

$$\left(p_k = \frac{\partial L}{\partial \dot{q}_k} \right)$$

The generalized momentum p_k associated with a cyclic or ignorable coordinate is a constant of motion of the system.

3.4 Hamiltonian function and conservation of energy:

Consider a general lagrangian L of a system given by

$$L = L\ (q_1,\ q_2, \ldots q_k \ldots\ldots, q_n, \dot{q}_1,\ \dot{q}_2,\ \ldots \dot{q}_k, \ldots \dot{q}_n, t)$$

And it also denoted by

$$L = L\ (q_k,\ \dot{q}_k,\ t)$$

The total time derivative of L is

$$\frac{dL}{dt} = \sum_k \frac{\partial L}{\partial q_k}\frac{dq_k}{dt} + \sum_k \frac{\partial L}{\partial \dot{q}_k}\frac{d\dot{q}_k}{dt} + \frac{\partial L}{\partial t} \qquad \text{------------(1)}$$

From lagrangian equations, we have

$$\frac{\partial L}{\partial q_k} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k} \right)$$

Substituting $\frac{\partial L}{\partial q_k}$ in equation (1), we get

$$\frac{dL}{dt} = \sum_k \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k} \right) + \sum_k \frac{\partial L}{\partial \dot{q}_k} \frac{d\dot{q}_k}{dt} + \frac{\partial L}{\partial t}$$

$$\frac{dL}{dt} = \sum_k \frac{d}{dt}\left(\dot{q}_k \frac{\partial L}{\partial \dot{q}_k} \right) + \frac{\partial L}{\partial t}$$

$$\frac{d}{dt}\left(\sum_k \dot{q}_k \frac{\partial L}{\partial \dot{q}_k} - L \right) = - \frac{\partial L}{\partial t} \qquad \text{-------------(2)}$$

The quantity in the bracket is sometimes called the energy function and is denoted by h:

$$h\,(q_1, q_2, \ldots\ldots, q_n, \dot{q}_1, \dot{q}_2, \ldots\ldots \dot{q}_n, t) = \sum_k \dot{q}_k \frac{\partial L}{\partial \dot{q}_k} - L \qquad \text{--------- (3)}$$

from equation (2) the total time derivative of h is

$$\frac{dh}{dt} = - \frac{\partial L}{\partial t}$$

If the lagrangian L does not depend on time t explicitly, then $\frac{\partial L}{\partial t} = 0$. So that

$$\frac{dh}{dt} = 0 \qquad \text{--------------------- (4)}$$

h - constant

when the lagrangian is not explicit function of time, the energy function is the constant of motion. It is one of the motion and is called Jacobi's integral.

We know that $\frac{\partial L}{\partial \dot{q}_k} = p_k$

Therefore, equation(1) can be written as,

$$\frac{d}{dt}\left(\sum_k p_k \dot{q}_k - L\right) = - \frac{\partial L}{\partial t} \qquad \text{----------------(5)}$$

The quantity in the bracket is called the **Hamiltonian function H.**

$$\text{i.e.,} \qquad \sum_k p_k \dot{q}_k - L = H \qquad \text{-----------------(6)}$$

Conservation of energy:

Hamiltonian takes a special form, if the system is conservative. i.e., the potential energy V is independent of velocity coordinates $\dot{q}_k$ and the transformation equations for coordinates do not contain time explicitly,

$$r_i = r_i \ (q_1, \ q_2, \ldots q_k \ldots\ldots, q_n)$$

For a conservative system $\frac{\partial V}{\partial \dot{q}_k} = 0.$

Therefore, $p_k = \frac{\partial L}{\partial \dot{q}_k} = \frac{\partial}{\partial \dot{q}_k}(T - V)$

$$= \frac{\partial T}{\partial \dot{q}_k}$$

So equation (6) is

$$H = \sum_k p_k \ \dot{q}_k - L$$

$$= \sum_k \dot{q}_k \frac{\partial T}{\partial \dot{q}_k} - L \qquad \text{-----------------(7)}$$

If r_i does not depend on time t explicitly, then the kinetic energy T is homogeneous quadratic function. It is easy to show that

$$\sum_k \dot{q}_k \frac{\partial T}{\partial \dot{q}_k} = 2T \qquad \text{-------------(8)}$$

In fact a natural conservative system neither T nor V contains any explicit time dependent and T is a homogeneous quadratic function of the time derivatives $\dot{q}_k$,

Hence equation(7) and (8)

$$H = 2T - L$$

$$= 2T - (T - V)$$

$$H = T + V = E, \text{ constant.}$$

Thus the Hamiltonian H represents the total energy of the system E and is conserved, provided the system is conservative and T is a homogeneous quadratic equation.

3.5 Canonical equations of motion:

Hamiltonian is a function of

$$H = H(q_k, \ p_k, \ t) \quad \text{------------(1)}$$

Differentiate the above equation,

$$dH = \frac{\partial H}{\partial q_k} \ dq_k + \frac{\partial H}{\partial p_k} \ dp_k + \frac{\partial H}{\partial t} \ dt$$

$$dH = \sum_{k=1}^{n} \frac{\partial H}{\partial q_k} \ dq_k + \frac{\partial H}{\partial p_k} \ dp_k + \frac{\partial H}{\partial t} \ dt \quad \text{-----------(2)}$$

we know that, $H = \sum_k p_k \ \dot{q}_k - L$

differentiate the above equation, we get

$$dH = \sum_k dp_k\, \dot{q}_k + \sum_k p_k\, d\dot{q}_k - dL \quad \text{----------------(3)}$$

$$dL = \sum_k \frac{\partial L}{\partial q_k}\, dq_k + \sum_k \frac{\partial L}{\partial \dot{q}_k}\, d\dot{q}_k + \frac{\partial L}{\partial t}\, dt \quad \text{----------------(4)}$$

where, $\frac{\partial L}{\partial q_k} = \dot{p}_k$, $\frac{\partial L}{\partial \dot{q}_k} = p_k$

the above values are substitute in equation(4), we get

$$dL = \sum_k \dot{p}_k\, dq_k + \sum_k p_k\, d\dot{q}_k + \frac{\partial L}{\partial t}\, dt \quad \text{-----------(5)}$$

substitute equation(5) in equation(3), we get

$$dH = \sum_k dp_k\, \dot{q}_k + \sum_k p_k\, d\dot{q}_k - \sum_k \dot{p}_k\, dq_k - \sum_k p_k\, d\dot{q}_k - \frac{\partial L}{\partial t}\, dt$$

$$dH = \sum_k dp_k\, \dot{q}_k - \sum_k \dot{p}_k\, dq_k - \frac{\partial L}{\partial t}\, dt \quad \text{----------------(6)}$$

compare the coefficient of equations(3) and equation(6), then we get

$$\boldsymbol{\dot{q}_k = \frac{\partial H}{\partial p_k}}$$

$$\boldsymbol{-\dot{p}_k = \frac{\partial H}{\partial q_k}}$$

$$\boldsymbol{\frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}}$$

The above equations are called the **Hamiltonian canonical equation.**

If q_k is a cyclic coordinate $\boldsymbol{\frac{\partial H}{\partial q_k} = 0}$.

From above equation $\boldsymbol{\dot{p}_k = 0}$,

$$p_k = \text{constant.}$$

3.6 Applications:

(i) One dimensional simple harmonic oscillator:

Consider a particle having simple harmonic motion. the particle executing simple harmonic motion, restoring force is directly proportional to displacements.

$$F \propto x$$

$$F = -kx \qquad \text{-------------(1)}$$

Where k is the force constant.

The potential energy $\qquad V = - \int F \, . \, dr$

$$V = - \int -kx \, . \, dx \qquad \text{---------------(2)}$$

$$V = \frac{kx^2}{2}$$

The kinetic energy $\qquad T = \frac{1}{2} m \dot{x}^2 \qquad$ -------------(3)

Lagrangian is given by

$$L = T - V \qquad \text{--------------------(4)}$$

Substitute equation (2) and (3) are substitute in equation (4)

$$L = \frac{1}{2} m \dot{x}^2 - \frac{kx^2}{2} \qquad \text{-----------(5)}$$

$$p_x = \frac{\partial L}{\partial \dot{x}}$$

$$p_x = \frac{\partial}{\partial \dot{x}} \left(\frac{1}{2} m \dot{x}^2 - \frac{kx^2}{2} \right)$$

$$p_x = m \dot{x}$$

$$\dot{x} = \frac{p_x}{m} \qquad \text{------------------(6)}$$

$$H = p_x \dot{x} - L \qquad \text{-------------------(7)}$$

Equation (5) and (6) are substitute in equation (7), we get

$$H = p_x \frac{p_x}{m} - \frac{1}{2} m \dot{x}^2 + \frac{kx^2}{2}$$

$$= \frac{P_x^2}{m} - \frac{1}{2} m \left(\frac{p_x}{m}\right)^2 + \frac{kx^2}{2}$$

$$= \frac{P_x^2}{m} + \frac{kx^2}{2}$$

Hamiltonian canonical equations are,

$$\dot{\boldsymbol{q}}_{\boldsymbol{k}} = \frac{\boldsymbol{\partial H}}{\boldsymbol{\partial p_k}}$$

$$-\dot{\boldsymbol{p}}_{\boldsymbol{k}} = \frac{\boldsymbol{\partial H}}{\boldsymbol{\partial q_k}}$$

$$\dot{x} = \frac{\partial H}{\partial p_k} = \frac{p_x}{m}$$

$$\dot{p}_x = - \frac{\partial H}{\partial x} = - kx \qquad \text{----------------(8)}$$

$$\dot{p}_x = \frac{dp_x}{dt} = \frac{d}{dt} (m \dot{x}) \qquad \text{------------------(9)}$$

Equating equations (8) and (9)

$$m \ddot{x} = - k x$$

$$m \ddot{x} + k x = 0 \qquad \text{--------------(10)}$$

This equation represents the equation of motion of a harmonic oscillator,

Divided by m in equation (10), we get

$$\ddot{x} + \frac{k}{m} x = 0$$

$$\ddot{x} + \omega^2 x = 0$$

Where $\omega^2 = \frac{k}{m}$,

$$\omega = \sqrt{\frac{k}{m}}$$

$$T = \frac{2\pi}{\omega}$$

$$= \frac{2\pi}{\sqrt{\frac{k}{m}}}$$

$$= 2\pi \sqrt{\frac{m}{k}}$$

$$\nu = \frac{1}{T}$$

$$= \frac{1}{2\pi \sqrt{\frac{m}{k}}}$$

$$= \frac{1}{2\pi} \sqrt{\frac{k}{m}}$$

(iii) motion of particle in a central force field:

All central forces are conservative in nature.

Kinetic energy $T = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2)$

Potential energy $V = - \frac{k}{r}$

$$L = T - V$$

Substitute T and V values,

$$L = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2) + \frac{k}{r}$$

Generalized momentum,

$$p_r = \frac{\partial L}{\partial \dot{r}}$$

$$= m \dot{r}$$

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}}$$

$$= m r^2 \dot{\theta}$$

From above equations we get,

$$\dot{r} = \frac{p_r}{m}$$

$$\dot{\theta} = \frac{p_\theta}{mr^2}$$

The Hamiltonian equation is

$$H = \sum_k p_k \dot{q}_k - L$$

$$H = p_r \dot{r} + p_\theta \dot{\theta} - \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2) + \frac{k}{r}$$

$$H = p_r \frac{p_r}{m} + p_\theta \frac{p_\theta}{mr^2} - \frac{1}{2} m \left(\frac{p^2{}_r}{m^2}\right) + \frac{1}{2} m r^2 \frac{(p_\theta)^2}{(mr^2)^2}) + \frac{k}{r}$$

$$H = \frac{p^2{}_r}{m} + \frac{p_\theta{}^2}{mr^2} - \frac{p^2{}_r}{2m} - \frac{p_\theta{}^2}{2mr^2} - \frac{k}{r}$$

$$H = \frac{p^2{}_r}{2m} + \frac{p_\theta{}^2}{2mr^2} - \frac{k}{r}$$

We know that $\dot{p}_k = -\frac{\partial H}{\partial q_k}$, $\dot{q}_k = -\frac{\partial H}{\partial p_k}$

$$\dot{r} = \frac{\partial H}{\partial p_r} = \frac{p_r}{m}$$

$$\dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{mr^2}$$

$$\dot{p}_r = -\frac{\partial H}{\partial r}$$

$$= \frac{\partial}{\partial r}\left(\frac{{p_\theta}^2}{2m} r^{-2} - k\, r^{-1}\right)$$

$$= \frac{{p_\theta}^2}{2m}\frac{(-2)}{r^3} + \frac{k}{r^2}$$

$$= -\frac{{p_\theta}^2}{mr^3} + \frac{k}{r^2}$$

$$= \frac{{p_\theta}^2}{mr^3} - \frac{k}{r^2}$$

$$\mathbf{m}\,\ddot{\boldsymbol{r}} - \frac{\boldsymbol{p_\theta}^2}{\boldsymbol{mr^3}} - \frac{\boldsymbol{k}}{\boldsymbol{r^2}} = \mathbf{0} \qquad \text{-----------(1)}$$

$$\dot{p}_\theta = -\frac{\partial H}{\partial \theta} = 0$$

$$\dot{p}_\theta = \frac{dp_\theta}{dt} = \frac{d}{dt}\left(mr^2\,\dot{\theta}\right)$$

$$\dot{\boldsymbol{p}}_{\boldsymbol{\theta}} = \boldsymbol{mr^2}\,\ddot{\boldsymbol{\theta}} + \mathbf{2\,m\,r}\,\dot{\boldsymbol{r}}\dot{\boldsymbol{\theta}} \qquad \text{---------------(2)}$$

The above two equations are the equation of motion of a particle under a central force.

UNIT IV: SMALL OSCILLATIONS

Stable and unstable equilibrium –Formulation of the problem: Lagrange's equations of motion for small oscillations – Properties of T, V and w– Normal co ordinates and normal frequencies of vibration– Free vibrations of a linear triatomic molecule.

4.1 Stable and unstable equilibrium:

A system is said to be in **stable equilibrium**, if a small displacement of the system from the rest position(by giving a small energy to it) results in a small bounded motion about the equilibrium position.

In case, small displacement of the system from the equilibrium position results in an unbounded motion, it is in an **unstable equilibrium.**

Further, if the system on displacement has no tendency to move about or away the equilibrium position, it is said to be **neutral equilibrium**.

Example:

An example of stable equilibrium is a pendulum in the rest position and that of an unstable equilibrium is an egg standing on one end. A coin placed flat anywhere on a stable is in neutral equilibrium.

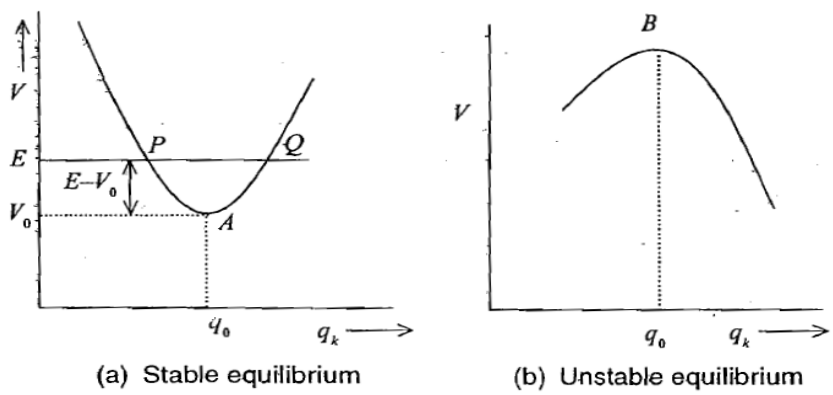

Fig. 4.1

A graph drawn between the potential energy of the system and a particular coordinate q_k, is called **potential energy curve.**

The position A and B, where F = - $\frac{\partial V}{\partial q}$ vanishes, are the position of equilibrium; potential energy V is minimum (V_0) at A is by displaced from A to Q by giving energy

(E - V_0) and left to itself, the system tries to come in the position of minimum potential energy.

Consequently, the potential energy will change to kinetic energy and at A the energy (E - V_0) will be purely in the kinetic form because of the conservation law.

This will be change again to potential form, when the system moves towards the position P and hence a bounded motion ensues about the equilibrium position A. Obviously, the position B of the maximum potential energy represents the unstable equilibrium because any energy given to the system at this position will result more and more kinetic energy when the system moves either left or right to it.

In this case, the system moves away from the equilibrium position. Incase of neutral equilibrium, the potential energy is independent of the coordinates and equilibrium occurs at any arbitrary value of that coordinate.

4.2 Lagrange's equations of motion for small oscillations:

The potential energy of a conservative system, specified by n generalized coordinates $q_1, q_2, q_3, \ldots\ldots\ldots, q_n$, is represented as

$$V = (q_1, q_2, \ldots\ldots\ldots, q_n) \qquad \text{------------ (1)}$$

When the system of the particles are small from the position of stable equilibrium. We denote the displacement of the generalized coordinates from equilibrium position u_i

$$\text{i.e.,} \qquad q_1 = q_i^0 + u_i \qquad \text{--------------- (2)}$$

Since, q_i^0 is fixed and u_i may be taken as new generalized coordinates of the motion. Expanding the potential energy about the position of equilibrium, we obtain

$$V(q_1, q_2, \ldots\ldots\ldots, q_n) = V (q_1^0 , q_2^0 , \ldots\ldots\ldots q_n^0) + \sum_{i=1}^{n}\left(\frac{\partial V}{\partial q_i}\right)_0 (q_i - q_i^0)$$

$$+ \frac{1}{2!}\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\frac{\partial^2 V}{\partial q_i \partial q_j}\right)(q_i - q_i^0) (q_j - q_j^0) + \ldots\ldots \text{------(3)}$$

In consequence of equilibrium $\left(\frac{\partial V}{\partial q_i}\right)_0 = 0$. First term in the expansion represents the potential energy in the equilibrium position and is constant for the system.

Assuming the potential energy in the equilibrium to be zero and writing $u_i = q_i - q_i^0$, and $u_j = q_j - q_j^0$, we get

$$V = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} V_{ij}\, u_i\, u_j \qquad \text{----------(4)}$$

Where $V_{ij} = \left(\frac{\partial^2 V}{\partial q_i \, \partial q_j}\right)$ = constant which is to be evaluated at $q_i = q_i^0$ and $q_j = q_j^0$.

The constant $V_{ij} = V_{ji}$ form a symmetric matrix V.

In equation(4) we retain the terms quadratic in the coordinates.

The kinetic energy of the system is given be

$$V = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} m_{ij} \ \dot{q}_i \ \dot{q}_j$$

$$= \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} m_{ij} \ \dot{u}_i \ \dot{u}_j \quad \text{--------------------(5)}$$

Because the generalized coordinates do not involve time explicitly and therefore kinetic energy is a homogeneous quadratic function of generalized velocities.

Thus, the coefficients are, in general, function of generalized coordinates and therefore expanding m_{ij} in Taylor's series, we get

$$m_{ij}\,(q_1, q_2, \ldots\ldots\ldots, q_n) = m_{ij}\,(q_1^0, q_2^0, \ldots\ldots\ldots q_n^0) + \sum_{k=1}^{n}\left(\frac{\partial m_{ij}}{\partial q_k}\right)_0 u_k + \ldots\ldots \text{-------- (6)}$$

In equation(5) the term is already quadratic in the $u_i{}'$ ' s we obtain the lowest non – vanishing approximation to T in quadratic form only be retaining the first term in the expansion. If the constant values of the function m_{ij} are denoted by T_{ij} , then the kinetic energy is

$$T = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} T_{ij} \ \dot{u}_i \ \dot{u}_j \quad \text{-------------- (7)}$$

Obviously, the constant T_{ij} are elements of symmetric matrix T. now, the lagrangian L (T -V) can be written as

$$L = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} \left(T_{ij} \ \dot{u}_i \ \dot{u}_j - V_{ij}\, u_i\, u_j \right) \quad \text{----------(8)}$$

Using u_i ' s as generalized coordinates, the lagrange's equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_i} = 0 \qquad \text{------------(9)}$$

Take the form

$$\sum_{j=1}^{n}(T_{ij}\, \ddot{u}_j + V_{ij}\, u_j)$$

4.3 Normal co-ordinates and normal frequencies of vibration:

Let us define

$$u_i = \sum_{k=1}^{n} a_{ik}\, Q_k \qquad \text{------------- (1)}$$

in terms of single column matrices

$$u = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \quad \text{and} \quad Q = \begin{bmatrix} Q_1 \\ Q_2 \\ \vdots \\ Q_n \end{bmatrix}$$

we have $u = A\,Q$ ---------------- (2)

the potential energy V can be written as

$$V = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} V_{ij}\; u_i\; u_j$$

$$= \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} u_i\; V_{ij}\; u_j$$

$$= \frac{1}{2}\,\bar{u}\, V\, u \qquad \text{------------------(3)}$$

Where $\bar{u}$ is the transpose of u or single row matrix.

From equation (2) $\bar{u} = \overline{A\;Q}$

$$= \bar{Q}\;\bar{A}$$

Therefore, $V = \frac{1}{2}\,\bar{Q}\;\bar{A}\;V\,A\;Q$ ------------------ (4)

The kinetic energy K similarly is,

$$T = \sum_i \sum_j \dot{u}_i\; T_{ij}\; \dot{u}_j$$

$$= \frac{1}{2}\,\bar{\dot{Q}}\;\bar{A}\;T\,A\;\dot{Q} \qquad \text{-------------------- (5)}$$

We know that $V\,a - \omega^2\;T\;a = 0$

Writing ${\omega_k}^2 = \lambda_k$

Therefore, equation (5) become,

$$\sum_{j=1}^{n}\left(V_{ij}\, a_{jk} - \lambda_k\, T_{ij}\, a_{jk}\right) = 0 \qquad \text{---------------- (6)}$$

The complex conjugate of this equation is

$$\sum_{j=1}^{n} \left(V_{ij}\, a_{il} * - \lambda_k\, T_{ij}\, a_{il} * \right) = 0 \qquad \text{---------------} \ (7)$$

As a_{ij} are real, we eliminate V_{ij} from equation(6) and (7) by multiplying the former by a_{il} and summing over I and the latter by a_{jk} and summing over j. thus

$$(\lambda_k - \lambda_l *) \sum_i \sum_j a_{jk}\; T_{ij}\; a_{il} \;=\; 0 \qquad \text{--------------} \ (8)$$

If all λ_k are distinct, i.e., ($\lambda_k - \lambda_l *$) is not zero, then

$$\sum_i \sum_j a_{jk}\; T_{ij}\; a_{il} \;=\; 0 \qquad \text{--------------} \ (9)$$

The two equations (8) and (9) can be combined into one by means of Kronecker delta symbol δ_{kl} , $i.e.$,

$$\sum_i \sum_j a_{jk}\; T_{ij}\; a_{il} \;=\; \delta_{kl} \qquad \text{-----------------} \ (10)$$

Equation(8) and (10) can be written as

$$\bar{A}\ \mathrm{T}\ \mathrm{A} \;=\; \mathrm{I} \qquad \text{--------------------} \ (11)$$

Writing $\lambda_l = \lambda_k\, \delta_{lk}$, we obtain equation(6)

$$\sum_{j=1}^{n} V_{ij}\, a_{jk} \;=\; \sum_{j=1}^{n} T_{ij}\, a_{jk}\;\; \lambda_k\, \delta_{lk} \qquad \text{---------}(12)$$

Which is in matrix notation,

$$\mathrm{V\ A} = \mathrm{T\ A}\ \lambda \qquad \text{---------------}(13)$$

Multiplying by $\bar{A}$

$$\bar{A}\ \mathrm{V\ A} = \ \bar{A}\ \mathrm{T\ A}\ \lambda \qquad \text{----------------}(14)$$

But $\qquad \bar{A}\ \mathrm{V\ A} \;=\; \lambda$

$$\bar{A}\ \mathrm{T\ A} \;=\; \mathrm{I} \qquad \text{----------------}(15)$$

In view of equation(15) equation(5) in obtained to be

$$\mathrm{V} \;=\; \tfrac{1}{2}\, \bar{Q}\ \lambda\ \mathrm{Q}$$

$$= \frac{1}{2}(Q_1, Q_2, \ldots\ldots Q_n)\begin{bmatrix}\lambda_1 & 0 & \ldots. & 0\\ 0 & \lambda_2 & \ldots. & 0\\ . & . & . & .\\ 0 & 0 & \ldots & \lambda_n\end{bmatrix}\begin{bmatrix}Q_1\\ Q_2\\ .\\ Q_n\end{bmatrix}$$

$$= \frac{1}{2}(\lambda_1 Q_1^2 + \lambda_2 Q_2^2 + \ldots\ldots\ldots. + \lambda_n Q_n^2)$$

$$= \frac{1}{2}\sum_{k=1}^{n} \lambda_k Q_k^2$$

$$= \frac{1}{2}\sum_{k=1}^{n} \omega_k^2 Q_k^2 \qquad \text{-------------(16)}$$

In view of equation(12) and (6) is obtained to be

$$T = \frac{1}{2}\bar{\dot{Q}}\, I\, \dot{Q}$$

$$= \frac{1}{2}\sum_{k=1}^{n} \dot{Q}_k^2 \qquad \text{---------------(17)}$$

Equation(16) and (17) that in the new coordinates, both the potential and kinetic energies are the sum of squares only without any cross terms.

Lagrangian $L = T - V$ is

$$L = \frac{1}{2}\sum_{k=1}^{n} \dot{Q}_k^2 - \frac{1}{2}\sum_{k=1}^{n} \omega_k^2 Q_k^2 \qquad \text{----------(18)}$$

The lagrangian equation is,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{Q}_k}\right) - \frac{\partial L}{\partial Q_k} = 0$$

For the new coordinates are,

$$\ddot{Q}_k + \omega_k^2 Q_k = 0 \qquad \text{-------------------- (19)}$$

Thus each new coordinates executes simple harmonic motion with a single frequency and therefore, $Q_1, Q_2, \ldots.., Q_n$ are called **normal coordinates**. The frequencies $\omega_1, \omega_2, \ldots\ldots \omega_n$ are referred as **normal frequencies.**

The solution of equation (19) is

$$Q_k = f_k \cos(\omega_k t + \varphi_k) \qquad \text{------------------- (20)}$$

From equation (19) and (18),

$$u_i = \sum_{k=1}^{n} a_{ik} Q_k$$

$$= \sum_{k=1}^{n} f_k\, a_{ik} \cos(\omega_k t + \varphi_k) \qquad \text{------------(21)}$$

Each normal coordinates corresponds to a vibration of the system with only one frequency and these component oscillations are called as the **normal modes of vibration.**

4.4 Free vibrations of a linear triatomic molecule:

Let us consider a linear triatomic molecule of the type AB_2 (CO_2) in which A atom in the middle and B atoms at the ends. The mass of A atom is M and that of each of the B atom is m. the interatomic force between A and B atom is approximated by elastic force of spring force constant k . the motion of the three atom is constrained along the line joining them.

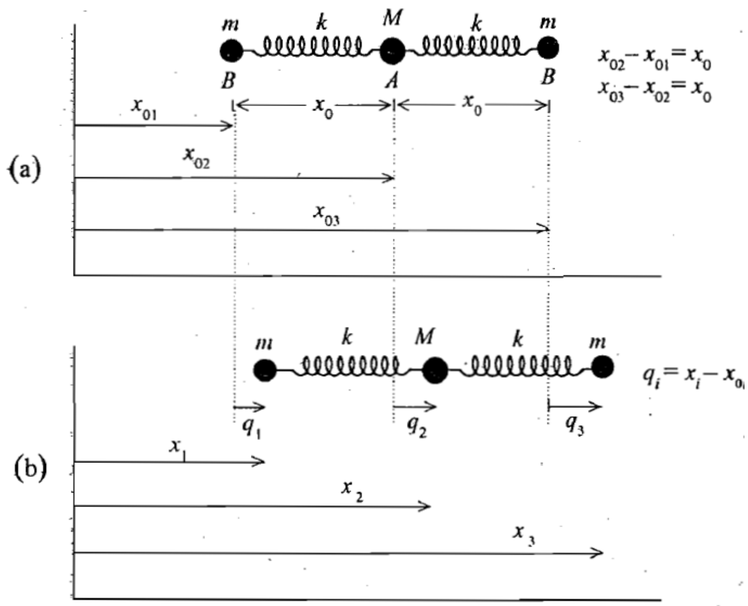

Fig. 4.2. Longitudinal oscillation of a linear symmetric triatomic molecule: (a) equilibrium configuration (b) configuration at any instant t

There are three coordinates marking the positions of three atoms on the line. If x_1, x_2 and x_3, are the positions of the three atoms at any instant from some arbitrary origin,

Then ,

$$V = \frac{1}{2} k (x_2 - x_1 - x_0)^2 + \frac{1}{2} k (x_3 - x_2 - x_0)^2$$

Where x_0 is the distance between any A and B atoms in the equilibrium configuration.

The generalized coordinate is defined as,

$$q_1 = x_1 - x_{01} \quad , q_2 = x_2 - x_{02} , \quad q_3 = x_3 - x_{03}$$

where,

$$x_{02} - x_{01} = x_{03} - x_{01} = x_0.$$

Then kinetic energy

$$T = \frac{1}{2} m (\dot{q}_1^2 + \dot{q}_3^2) + \frac{1}{2} M \dot{q}_2^2$$

Potential energy

$$V = \frac{1}{2} k (q_2 - q_1)^2 + \frac{1}{2} k (q_3 - q_2)^2$$

T and V matrices are

$$T = \begin{pmatrix} m & 0 & 0 \\ 0 & M & 0 \\ 0 & 0 & m \end{pmatrix} \quad \text{amd} \quad V = \begin{pmatrix} k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & k \end{pmatrix} \quad \text{---------------(1)}$$

The secular equation is,

$$| V - \omega^2 I | = \begin{pmatrix} k - m\omega^2 & -k & 0 \\ -k & 2k - M\omega^2 & -k \\ 0 & -k & k - M\omega^2 \end{pmatrix} = 0 \quad \text{--------(2)}$$

$$\omega^2 (k - m\omega^2) (k (M + 2m) - \omega^2 Mm) = 0$$

The solution of the above equation is

$$\omega_1 = 0, \ \omega_2 = \sqrt{\frac{k}{m}}, \text{ and } \omega_3 = \sqrt{\frac{k(1 + \frac{2m}{M})}{m}} \text{-------------(3)}$$

To determine the eigen vectors, we use the equation

$$(V - \omega_k{}^2 T) a_k = \begin{pmatrix} k - m\omega^2 & -k & 0 \\ -k & 2k - M\omega^2 & -k \\ 0 & -k & k - M\omega^2 \end{pmatrix} \begin{pmatrix} a_{1k} \\ a_{2k} \\ a_{3k} \end{pmatrix} = 0$$

The eigen vectors for the three modes of vibrations.

(a) $\omega_1 = 0$

$$\begin{pmatrix} k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & k \end{pmatrix} \begin{pmatrix} a_{11} \\ a_{22} \\ a_{33} \end{pmatrix} = 0$$

$$a_{11} - a_{21} = 0,$$

$$-a_{11} + 2a_{21} - a_{31} = 0,$$

$$-a_{21} + a_{31} = 0$$

Or $a_{11} = a_{21} = a_{31} = \alpha$

Therefore the eigen vectors are given by,

$$a_1 = \begin{pmatrix} \alpha \\ \alpha \\ \alpha \end{pmatrix} \qquad \text{---------------------(4)}$$

(b) $\omega_2 = \sqrt{\frac{k}{m}}$

$$\begin{pmatrix} 0 & -k & 0 \\ -k & 2k - \frac{Mk}{m} & -k \\ 0 & -k & 0 \end{pmatrix} \begin{pmatrix} a_{12} \\ a_{22} \\ a_{32} \end{pmatrix} = 0$$

$$a_{22} = 0,$$

$$-a_{12} - a_{32} = 0$$

Therefore, $a_{22} = 0, \quad a_{12} = -a_{32} = \beta$

The eigen vectors are given by,

$$a_2 = \begin{pmatrix} \beta \\ 0 \\ -\beta \end{pmatrix} \qquad \text{---------------------(5)}$$

(c) $\omega_3 = \sqrt{\frac{k(1 + \frac{2m}{M})}{m}}$

$$\begin{pmatrix} -\frac{2mk}{M} & -k & 0 \\ -k & -\frac{Mk}{m} & -k \\ 0 & -k & -\frac{2mk}{M} \end{pmatrix} \begin{pmatrix} a_{13} \\ a_{23} \\ a_{33} \end{pmatrix} = 0$$

$$\frac{Mk}{m} a_{13} + a_{23} = 0,$$

$$a_{13} + \frac{m}{M} a_{23} + a_{33} = 0,$$

Therefore, $a_{23} = a_{23} = \gamma$, and $a_{23} = -\frac{2m}{M}\gamma$

The eigen vectors are given by,

$$a_3 = \begin{pmatrix} \gamma \\ -\frac{2m}{M}\gamma \\ \gamma \end{pmatrix} \qquad \text{-----------------(6)}$$

now, A matrix is

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

$$= \begin{pmatrix} \alpha & \beta & \gamma \\ \alpha & 0 & -\frac{2m}{M}\gamma \\ \alpha & -\beta & \gamma \end{pmatrix} \qquad \text{-------------------(7)}$$

We know the condition,

$$\bar{A}\ T\ A = I$$

$$\begin{pmatrix} \alpha & \alpha & \alpha \\ \beta & 0 & -\beta \\ \gamma & -\frac{2m}{M}\gamma & \gamma \end{pmatrix}\begin{pmatrix} m & 0 & 0 \\ 0 & M & 0 \\ 0 & 0 & m \end{pmatrix}\begin{pmatrix} \alpha & \beta & \gamma \\ \alpha & 0 & -\frac{2m}{M}\gamma \\ \alpha & -\beta & \gamma \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} \alpha^2\,(2m + M) & 0 & 0 \\ 0 & 2\,\beta^2 & 0 \\ 0 & 0 & 2\gamma^2\, m(1 + \frac{2m}{M}) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Thus, $\alpha = \frac{1}{\sqrt{2m + M}}$

$$\beta = \frac{1}{\sqrt{2m}}$$

$$\gamma = \frac{1}{\sqrt{2m\,(1 + \frac{2m}{M})}}$$

The eigen vectors are

$$a_1 = \frac{1}{\sqrt{2m + M}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$a_2 = \frac{1}{\sqrt{2m}}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

$$a_3 = \frac{1}{\sqrt{2m\,(1+\frac{2m}{M})}} \begin{pmatrix} 1 \\ -\frac{2m}{M} \\ 1 \end{pmatrix} \qquad \text{------------(8)}$$

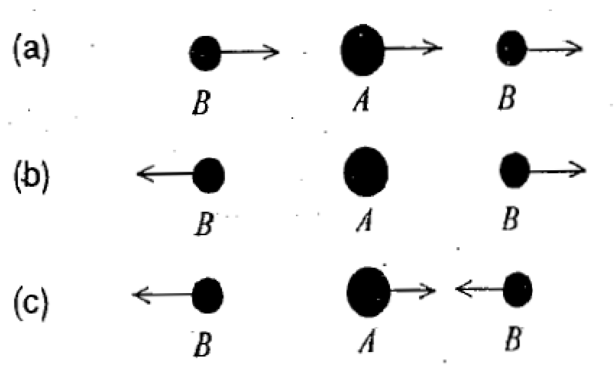

Fig. 4.3. Longitudinal normal modes of the triatomic molecule:

(a) Mode 1: all the three atoms are displaced equally in the same direction

(b) Mode 2: A atom does not vibrate and B atoms oscillate with equal amplitude but in opposite directions.

(c) Mode 3: B atoms vibrate in phase with equal amplitudes and the middle atom A vibrates in opposite phase with different amplitude.

Case 1:

$a_{11} = a_{21} = a_{31}$ means the displacements of three atoms are the same in the same direction. This is what expected from translatory motion.

Case 2:

$a_{22} = 0$ and $a_{12} = - a_{33}$ implies that in this mode, the middle atom does not vibrate and the end atoms(B) oscillate with equal amplitudes but in opposite direction.

Case 3:

$a_{13} = a_{33} = \gamma$, and $a_{23} = -\left(\frac{2m}{M}\right)\gamma$. it show that the end atoms oscillate in phase with equal amplitude, while the central atom vibrates in opposite phase with different amplitude.

The generalized coordinate q_1, q_2 and q_3 are related to the normal coordinates Q_1, Q_2, and Q_3 by using the relation,

$$q_i = \sum_{k=1}^{3} a_{ik}\, Q_k$$

therefore,

$$\begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} = \begin{pmatrix} \alpha & \beta & \gamma \\ \alpha & 0 & \pm\frac{2m}{M}\gamma \\ \alpha & \pm\beta & \gamma \end{pmatrix} \begin{pmatrix} Q_1 \\ Q_2 \\ Q_3 \end{pmatrix} \qquad \text{--------------(9)}$$

The normal coordinate Q_1 oscillates with frequency $\omega_1 = 0$, Q_2 with $\omega_2 = \sqrt{\frac{k}{m}}$ and Q_3 with $\omega_3 = \frac{1}{\sqrt{2m\,(1+\frac{2m}{M})}}$. So that

$$Q_1 = f_1 \cos(\omega_1 t + \varphi_1),$$

$$Q_2 = f_2 \cos(\omega_2 t + \varphi_2)$$

$$Q_3 = f_3 \cos(\omega_3 t + \varphi_3)$$

Thus $q_1 = \alpha\ f_1 \cos(\omega_1 t + \varphi_1) + \beta\ f_2 \cos(\omega_2 t + \varphi_2) + \gamma\ f_3 \cos(\omega_3 t + \varphi_3)$

Or $x_1 = A \cos(\omega_1 t + \varphi_1) + B \cos(\omega_2 t + \varphi_2) + C \cos(\omega_3 t + \varphi_3)$

$\omega_t = 0$, therefore,

$$x_1 = A' + B \cos(\omega_2 t + \varphi_2) + C \cos(\omega_3 t + \varphi_3) + x_{01}$$

$$x_2 = A' - \frac{2m}{M} C \cos(\omega_3 t + \varphi_3) + x_{02}$$

$$x_3 = A' - B \cos(\omega_2 t + \varphi_2) + C \cos(\omega_3 t + \varphi_3) + x_{03}$$

where A’ represent the constant corresponding to rigid translation and x_{0i} represent the equilibrium position of an atom.

UNIT V: RELATIVITY

Inertial and non-inertial frames – Lorentz transformation equations – length contraction and time dilation–relativistic addition of velocities – Einstein's mass-energy relation Minkowski's space–four vectors – position, velocity, momentum, acceleration and force in four vector notation and their transformations.

5.1 Inertial and non-inertial frames:

Event: an event is something that happens at a particular point in space and at a particular instant of time, independent of the reference frame. Which we may use to described it.

A collision between two particles, an explosion of bomb or star and a sudden flash of light are the examples of event.

Observer: an observer is a person or equipment meant to observe and take measurement about the event. The observer is supposed to have with him scale, clock and other needful things to observe that event.

Inertial frame: an inertial frame is defines as reference frame in which the law of inertial holds true. i.e., Newton's first law. such a frame is also called un – accelerated frame. E.g. a distant star can be selected as slandered inertial frame of reference.

Non – inertial frame: it is defined as a set of coordinates moving with acceleration relative to some other frame in which the law of inertia does not hold true. It is an accelerated frame. E.g. applications of brakes to a moving train makes it an accelerated(decelerated) frame. So it becomes a non – inertial frame.

5.2 Lorentz transformation equations:

Consider two frames of reference s and s' . as shown in figure s is fixed and s' is moving along the direction of x – axis with a constant velocity.

After time t the frame of reference s' has moved a distance xx' = vt.

For the point P in space the coordinates are (x, y, z) with the reference to the frame s and (x', y', z') with reference to the fame s'

According to Galilean transformation equation

$$x' = x - vt$$

$$y' = y$$

$$z' = z$$

$$t' = t \qquad\qquad \text{--------------------(1)}$$

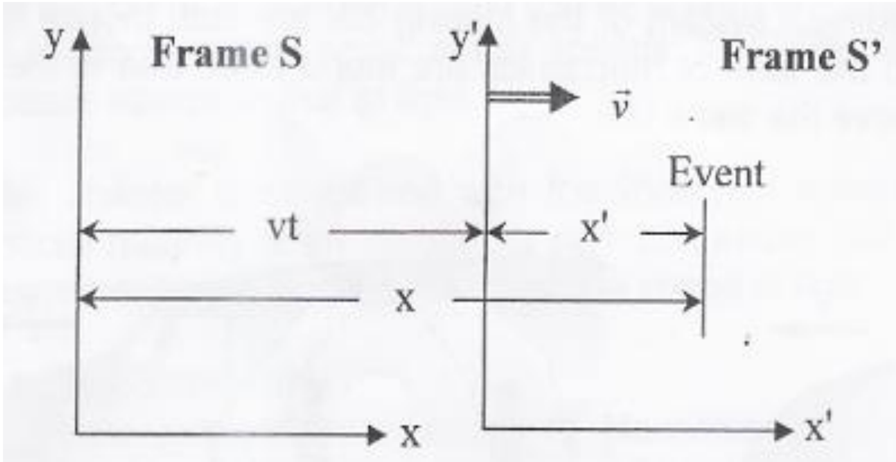

Fig. 5.1

Differentiate the equation (1)

$$\frac{dx'}{dt'} = \frac{dx}{dt} - v$$

$$c' = c - v$$

This equation says if a person is moving in a spaceship the speed of the passing light will be (c - v)

But according to the postulates of the special theory of relativity the velocity of light remains constant in free space.

This suggests that the Galilean transformations are not in accordance with the special theory of relativity. So the need for the new transformation equations is there.

However, the equation $x' = x - vt$ is in accordance with the ordinary laws of mechanics. So the new transformation for the x coordinates must be similar to this equation. The simplest possible for of this can be

$$x' = k(x - vt) \qquad\qquad \text{-------------------(2)}$$

where k depends only on the value of v and doen't depend upon the values of x and t. the above equation is linear and x' has only one value for given value of x.

According to the first postulate of the special theory of relativity observation made in the frame of reference s' must be identical to those made in s expect for a change in the sign of v and having the same value for the constant of proportionality k .

$$x = k (x' + v t') \qquad \text{-------------------(3)}$$

Since the relative motion of s and s' is combined to only x

$$y' = y$$

$$z' = z$$

$$t' = t$$

the value of x' from equation(2) can be substituted in equation(3)

$$x = k (k(x - vt) + vt')$$

$$x = k^2 (x – vt) + k v t'$$

$$k v t' = x - k^2 (x – vt)$$

$$t' = \frac{x- k^2 x + k^2 v t}{kv}$$

$$t' = \frac{x- k^2 x}{kv} + \frac{k^2 v t}{kv}$$

$$= \frac{x (1- k^2)}{kv} + \frac{k^2 v t}{kv}$$

$$= k t + \frac{x (1- k^2)}{kv} \qquad \text{--------------(4)}$$

To find the value k, consider two reference frames s and s'. the spaceship in reference frame s measure the time t and the spaceship in reference frame s' measure timt t'.

$$x = ct \qquad \text{------------------(5)}$$

$$x' = c t' \qquad \text{-------------------(6)}$$

substituting the value of x' and t' from equation (2) and (4) in equation (6)

$$k (x – vt) = c \left(k t + \frac{x (1- k^2)}{kv} \right)$$

$$k x – k v t = c k t + c \frac{x (1- k^2)}{kv}$$

$$k x - c \frac{x (1- k^2)}{kv} = c k t + k v t$$

$$x \left(k - c \frac{(1- k^2)}{kv} \right) = c k t (1 + \frac{v}{c})$$

$$x = \frac{c k t (1+ \frac{v}{c})}{k - c\frac{(1- k^2)}{kv}}$$

$$x = \frac{c\,k\,t\,(1+\frac{v}{c})}{k(1 - c\frac{\left(\frac{1}{k^2}-1\right)}{v})}$$

$$x = c\,t\,\frac{(1+\frac{v}{c})}{(1 - c\frac{\left(\frac{1}{k^2}-1\right)}{v})} \qquad \text{-------------------(7)}$$

then substituting the value of x from equation (5) into (7) we get

$$c\,t = c\,t\,\frac{(1+\frac{v}{c})}{(1 - c\frac{\left(\frac{1}{k^2}-1\right)}{v})}$$

$$1 - c\frac{\left(\frac{1}{k^2}-1\right)}{v} = 1 + \frac{v}{c}$$

$$-\left(\frac{c}{v}\right)\left(\frac{1}{k^2} - 1\right) = \frac{v}{c}$$

$$1 - \frac{1}{k^2} = \frac{v^2}{c^2}$$

$$1 - \frac{v^2}{c^2} = \frac{1}{k^2}$$

$$k^2 = \frac{1}{1-\frac{v^2}{c^2}} \qquad \text{-------------------(8)}$$

$$k^2 = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \qquad \text{---------------------(9)}$$

the value of k when substituted in equation (2) we get

$$x' = k\,(x - vt)$$

$$\mathbf{x' = \frac{(x-vt)}{\sqrt{1-\frac{v^2}{c^2}}}} \qquad \text{---------------------(10)}$$

$$y' = y \qquad \text{----------------------(11)}$$

$$z' = z \qquad \text{---------------------(12)}$$

now we can rewrite the equation (4)

$$t' = k\,t + \frac{x\,(1-k^2)}{kv}$$

$$t' = k\,t + \frac{x}{kv} - k\left(\frac{x}{v}\right)$$

$$t' = k\,t + \frac{x}{v}\left(\frac{1}{k} - k\right)$$

$$t' = k\,t + k\,\frac{x}{v}\left(\frac{1}{k^2} - 1\right)$$

$$t' = k\left(t + \frac{x}{v}\left(\frac{1}{k^2} - 1\right)\right)$$

$$t' = \frac{t + \frac{x}{v}\left(1 - \frac{v^2}{c^2} - 1\right)}{\sqrt{1 - \frac{v^2}{c^2}}}$$

then substituting the value of k and k^2 in above equation from equation(9) and (8), we get

$$t' = k\left(t + \frac{x}{v}\left(\frac{1}{k^2} - 1\right)\right) = \frac{t + \frac{x}{v}\left(1 - \frac{v^2}{c^2} - 1\right)}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$\mathbf{t' = \frac{t + \frac{x\,v}{t^2}\,)}{\sqrt{1 - \frac{v^2}{c^2}}}} \qquad \text{---------------(13)}$$

the equations (10), (11), (12), (13) are called **Lorentz transformation equations**.

These equations give the conversions for the measurements of time and space made in stationary frame s to s'.

5.3 Consequences of Lorentz transformations:

(i) Length contraction

(ii) Time dilation

(iii) Addition of velocities

5.3.1 Length contraction:

Measurement of space and time are not absolute but depend on the relative motion of the observer and the observed objects.

Consider a rod of length L_0 parallel to the x – axis and having co – ordinates x_1 and x_2 in the frame s.

An observer in the reference frame s measure the length of the rod as $L_0 = x_2 - x_1$.

Also consider a second reference frame' s moving with a velocity v along x- axis with respect to the reference frame s.

An observer in the reference frame s' measures the length of the rod as $L = x_2' - x_1'$

The relation between x_1 and x_1' and also between x_2 and x_2' according to inverse Lorents Transformation will be

$$x_1 = \frac{x_1' + v\,t'}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$x_2 = \frac{x_2' + v\,t'}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$L_0 = x_2 - x_1$$

$$L_0 = \frac{x_2' + v\,t'}{\sqrt{1 - \frac{v^2}{c^2}}} - \frac{x_1' + v\,t'}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$L_0 = \frac{x_2' - x_1'}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$L_0 = \frac{L}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$L = L_0\sqrt{1 - \frac{v^2}{c^2}}$$

This equation shows that the length of stationary object with respect to an observer in motion appears to be shorter than length measured by an observer at rest.

Similarly when the object is in motion with respect to a stationary observer, again the object appears to be shortened in length.

This relativistic result is true for both the cases, i.e., whether object is in motion or the observer is in motion the object appears to be contracted or shortened in length. This phenomenon is called Lorentz - Fitzeralds contractions.

Lorentz - Fitzeralds contractions is appreciable only when the velocity v is comparable to the velocity of light c.

Let consider a rod of length L moving with a velocity which is equal to 0.6c. then its length as measured from another frame is given by L_0. Here we have $v = 0.6c$

$$L = L_0 \sqrt{1 - \frac{v^2}{c^2}}$$

$$L = L_0 \sqrt{1 - 0.36}$$

$$L = 0.8\ L_0$$

The contraction in length = $L_0 - L = 0.2\ L_0$

Instead if the velocity of the body is negligible small as compared to c, the contraction in length is also negligible i.e., compared to c, the contraction in length is also negligible. i.e.,

$$L_0 = L$$

5.3.2 Time dilation:

According to the special theory of relativity the time intervals are also affected by the relative motion between two frames of reference.

Let's consider two frames of references A and B such that B is moving along x axis with respect to A with a constant velocity v, the duration of an event taking place at a point in space is measured from both the frames of reference.

The observers in both the frames measure the time instants of beginning of the event and then the time instants of the ending of the event.

Suppose, the event begins time t_1 in frame A and at time t_1' in frame B. the event ends at time t_2 in frame A and at time t_2' in frame B. if the time interval measured in a is t and in B it is t_0, then we have,

$$t_0 = t_2' - t_1' \qquad \text{--------------------(1)}$$

and

$$t = t_2 - t_1 \qquad \text{------------------(2)}$$

According to the Inverse Lorentz transformation equation,

$$t_1 = \frac{t_1' + x'\frac{v}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}} \qquad \text{and}$$

$$t_2 = \frac{t_2' + x'\frac{v}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}} \qquad \text{----------------(3)}$$

equation (2) become

$t = t_2 - t_1$

$$t = \frac{t_2' + x'\frac{v}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}} - \frac{t_1' + x'\frac{v}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$t = \frac{t_2' + t_1'}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$t = \frac{t_0}{\sqrt{1-\frac{v^2}{c^2}}} \qquad \text{-------------------- (4)}$$

From equation(4) we can say that the interval of time t_0 measured in the moving frame B is smaller than that in stationary frame A.

It shows that the moving clock appears to go slower than the stationary clock.

So to a moving observer time appears to be expanded. This phenomenon is called time dilation.

5.3.3 Addition of velocities:

Let the coordinates of a particle in frame s be (x, y, z, t) and in frame s' (x', y', z', t), then the components of its velocity in two frames can be written as

$$u_x = \frac{dx}{dt}, \quad u_y = \frac{dy}{dt}, \quad \text{and} \quad u_z = \frac{dz}{dt} \qquad \text{in s}$$

$$u_x' = \frac{dx'}{dt}, \quad u_y' = \frac{dy'}{dt}, \quad \text{and} \quad u_z' = \frac{dz'}{dt} \qquad \text{in s'}$$

According to the inverse Lorentz transformation

$$x = \gamma(x' + vt'),$$
$$y = y'$$
$$z = z'$$
$$t = \gamma\left(t' + \frac{v}{c^2}x'\right)$$

therefore, $dx = \gamma(dx' + v\,dt'),$

$$dy = y'$$
$$dz = dz'$$
$$dt = \gamma\left(dt' + \frac{v}{c^2}dx'\right)$$

then,

$$u_x = \frac{dx}{dt}$$

$$= \frac{\gamma\,(\,dx' + v\,dt'\,)}{\gamma\,(\,dt' + \frac{v}{c^2}\,dx'\,)}$$

$$= \frac{\frac{dx'}{dt'} + v}{1 + \frac{dx'}{dt'}\frac{v}{c^2}}$$

$$= \frac{u_x' + v}{1 + u_x'\frac{v}{c^2}}$$

Similarly,

$$u_y = \frac{u_y'}{\gamma\,(\,1 + u_x'\frac{v}{c^2}\,)}$$

$$u_z = \frac{u_z'}{\gamma\,(\,1 + u_x'\frac{v}{c^2}\,)}$$

this is the relativistic law of addition of velocities while in classical mechanics

$u_x = u_x' + v$, $u_y = u_y'$, and $u_z = u_z'$

when v is less than the speed of light c.

if we take the Lorentz transformation, we can prove that

$$u_x' = \frac{u_x + v}{1 + u_x\frac{v}{c^2}}$$

$$u_y' = \frac{u_y}{1 + u_x\frac{v}{c^2}}$$

$$u_z' = \frac{u_z}{1 + u_x\frac{v}{c^2}}$$

In case a particle (as photon) is moving with a velocity c in the frame s' and s is moving with velocity c relative to s along positive x -axis direction,

$$u_x = \frac{c + c}{1 - \frac{cc}{c^2}} \qquad = \qquad c$$

because $u_x' = c$, $v = c$.

5.4 Einstein's mass-energy relation:

In relativity, work done by a force = $\int \vec{F}.\,\overrightarrow{ds}$

Relativistic kinetic energy k is given by

$$k = \int_{u=0}^{u=d} F\,ds$$

$$= \int_{u=0}^{u=d} \frac{d\,(m\,u)}{dt}\,\frac{ds}{dt}\,dt$$

Since $\frac{ds}{dt} = u$

$$k = \int_{u=0}^{u=d} u\,\frac{d\,(m\,u)}{dt}\,dt$$

$$= \int_{u=0}^{u=d} u\,d\,(m\,u)$$

The relativistic formula is

$$m = \frac{m_0}{\sqrt{1-\frac{u^2}{c^2}}}$$

$$m^2 = \frac{{m_0}^2}{\frac{c^2-u^2}{c^2}}$$

$$m^2\,(c^2 - u^2) = {m_0}^2\,c^2$$

$$m^2 c^2 - m^2 u^2) = {m_0}^2\,c^2$$

taking differentiation on both sides

$$2mc^2\,dm - m^2\,2\,u\,du - u^2\,2m\,dm = 0$$

Dividing the equation by 2m, we get

$$\frac{2\mathrm{m}\,c^2\,\mathrm{dm}}{2m} - \frac{m^2\,2\,\mathrm{u\,du}}{2m} - \frac{u^2\,2\mathrm{m\,dm}}{2m} = 0$$

$$\frac{2\mathrm{m}\,c^2\,\mathrm{dm}}{2m} = \frac{m^2\,2\,\mathrm{u\,du}}{2m} + \frac{u^2\,2\mathrm{m\,dm}}{2m}$$

$$u\,d(m\,u) = m\,u\,du + u^2\,dm = c^2\,dm$$

$$k = \int_{u=0}^{u=d} u \frac{d(m u)}{dt} dt$$

$$= \int_{u=0}^{u=d} u\, d(m u)$$

$$= c^2 \int_{u=0}^{u=d} dm$$

$$= m c^2 - m_0 c^2$$

$$= \frac{m_0}{\sqrt{1-\frac{u^2}{c^2}}} (c^2 - m_0 c^2)$$

$$= m_0 c^2 \left(\frac{1}{\sqrt{1-\frac{u^2}{c^2}}} - 1 \right)$$

$$= m_0 c^2 \left(\left(1 - \frac{u^2}{c^2}\right)^{-1/2} - 1 \right)$$

Using the Binomial expansion of u/c all get

$$k = m_0 c^2 \left(1 + \frac{1}{2}\left(\frac{u}{c}\right)^2 + \frac{3}{8}\left(\frac{u}{c}\right)^4 + \ldots\ldots\ldots \right)$$

$$k = m_0 c^2 \left(1 + \frac{1}{2}\left(\frac{u}{c}\right)^2 \right)$$

$$k = m_0 c^2 \frac{1}{2}\left(\frac{u}{c}\right)^2$$

$$k = \frac{1}{2} m_0 u^2$$

which is the classical result obtained by neglecting higher order terms

$$E = m_0 c^2 + \frac{1}{2} m_0 u^2$$

This equation represents the equivalence of mass and energy.

5.5 Minkowski's space:

Minkowski considered a four dimensional cartesian space in which the position is specified by three coordinates x, y, z and the time is referred by a fourth coordinate *ict.*

If we write $x_1 = x$, $x_2 = y$, $x_3 = z$ and $x_4 = ict$, then an event is represented by the position vector (x_1, x_2, x_3, x_4) in this four dimensional space. Of course the fourth dimension , referring to time, is imaginary. This four dimensional space is called **Minkowski** or **world space**.

It is also referred as **space-time continuum** and sometimes briefly as **four-space**.

The square of the magnitude of the position vector in such a four-space has the form

$$s^2 = x_1^2 + x_2^2 + x_3^2 + x_4^2$$

$$= x^2 + y^2 + z^2 - c^2t^2 \qquad \text{--------------(1)}$$

Lorentz transformations are designed so that the speed of light remains constant in S and S' inertial frames (S' is moving with constant velocity v relative to S) and this condition is equivalent to require that the position vector in the four-space is held invariant under the transformations, i.e.,

$$s^2 = x'^2 + y'^2 + z'^2 - c^2 t'^2$$

$$= x^2 + y^2 + z^2 - c^2t^2$$

$$s^2 = x'^2_1 + x'^2_2 + x'^2_3 + x'^2_4$$

$$= x_1{}^2 + x_2{}^2 + x_3{}^2 + x_4{}^2$$

$$s^2 = \sum_{\mu=1}^{4} x'^2_\mu$$

$$= \sum_{\mu=1}^{4} x_\mu{}^2 \qquad \text{------------------- (2)}$$

This equation is analogous to the distance – preserving orthogonal transformation for rotation from one frame of reference to another in three dimensional space.

Thus the coordinates x_1 , x_2, x_3, and x_4 chosen above, for an orthogonal coordinate system in four dimension and equation (2) implies that the transformations which we are seeking, correspond to a rotation in a four dimensional space. In fact, these orthogonal transformations in the four – dimensional Minkowski space are the Lorentz transformations.

5.6 Four vectors :

A vector in four dimensional Minkowski space is called a four – vector. Its components transform from one frame to another similar to Lorenz transformation.

An event in four dimensional space is represented by a world point(x_1 , x_2, x_3, x_4). The Lorentz transformations from s – frame to s' frame correspond to orthogonal transformations in the four-space and are represented as

$$x_\mu{}' = \sum_{v=1}^{4} a_{\mu v}\, x_v$$

$$\begin{pmatrix} x_1{}' \\ x_2{}' \\ x_3{}' \\ x_4{}' \end{pmatrix} = \begin{pmatrix} \gamma & 0 & 0 & i\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -i\beta\gamma & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \qquad \text{-----------------(1)}$$

With the condition

$$\sum_{\mu=1}^{4} x'^2_\mu \quad = \quad \sum_{\mu=1}^{4} x_\mu{}^2 \qquad \text{-----------------(2)}$$

We may represent the position vector of a world point b

$$x_\mu = (x_1 , x_2, x_3, x_4) = (r, ict) \qquad \text{---------------------(3)}$$

Where (x_1 , x_2, x_3, x_4) and (x, y, z) represent the position vector r of a point in three dimensional space and x_4 = ict or $x_4 = iT$. r (= x, y, z) is the space part and ict is the time part of the four dimensional position vector x_μ.

A four vector A_μ is a vector in four dimensional space with components A_1 , A_2, A_3 and A_4 and is represented as

$$A_\mu = (A_1 , A_2, A_3 , A_4)$$

$$= (A, iA_t) \qquad \text{--------------------(4)}$$

Where A (= A_1 , A_2, A_3) is the space component and A_4 (= i A_t) is the time component. These components transform from s frame to s' frame similar to Lorents transformation.i.e.,

$$\begin{pmatrix} A_1{}' \\ A_2{}' \\ A_3{}' \\ A_4{}' \end{pmatrix} = \begin{pmatrix} \gamma & 0 & 0 & i\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -i\beta\gamma & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} A \\ A_2 \\ A_3 \\ A_4 \end{pmatrix} \qquad \text{------------(5)}$$

$$A_\mu{}' = \sum_{v=1}^{4} a_{\mu v}\, A'_v$$

$$A_1{}' = \gamma (A_l + i\beta A_4)$$

$$A_2{}' = A_2$$

$$A_3{}' = A_3$$

$$A_4' = \gamma (A_4 - i\beta A_1) \qquad \text{-----------------(6)}$$

Examples of four vector:

5.7 Position four – vector (x_μ):

It is expressed as

$$x_\mu = (x_1 , x_2, x_3, x_4)$$

$$= (r, ict)$$

5.8 Velocity four – vector (u_μ):

The components of the velocity four – vector u_μ are defined

$$u_1 = \frac{dx_1}{dt} = \frac{dx_1}{dt}\frac{dt}{d\tau}$$

$$= \frac{dx}{dt} \frac{1}{\sqrt{1-\frac{u^2}{c^2}}} = \frac{u_x}{\sqrt{1-\frac{u^2}{c^2}}}$$

$$u_2 = \frac{dx_2}{dt} = \frac{dx_2}{dt}\frac{dt}{d\tau}$$

$$= \frac{dy}{dt} \frac{1}{\sqrt{1-\frac{u^2}{c^2}}} = \frac{u_y}{\sqrt{1-\frac{u^2}{c^2}}}$$

$$u_3 = \frac{dx_3}{dt} = \frac{dx_3}{dt}\frac{dt}{d\tau}$$

$$= \frac{dz}{dt} \frac{1}{\sqrt{1-\frac{u^2}{c^2}}} = \frac{u_z}{\sqrt{1-\frac{u^2}{c^2}}}$$

$$u_4 = \frac{dx_4}{dt} = \frac{d(ict)}{dt}\frac{dt}{d\tau}$$

$$= \frac{ic}{\sqrt{1-\frac{u^2}{c^2}}}$$

Where $\frac{dt}{d\tau} = \frac{1}{\sqrt{1-\frac{u^2}{c^2}}}$

Hence $u_\mu = (u_1, u_2, u_3, u_4)$

$$u_\mu = \left(\frac{u_x}{\sqrt{1-\frac{u^2}{c^2}}}, \frac{u_y}{\sqrt{1-\frac{u^2}{c^2}}}, \frac{u_z}{\sqrt{1-\frac{u^2}{c^2}}}, \frac{ic}{\sqrt{1-\frac{u^2}{c^2}}} \right)$$

$$u_\mu = \left(\frac{u}{\sqrt{1-\frac{u^2}{c^2}}}, \frac{ic}{\sqrt{1-\frac{u^2}{c^2}}} \right)$$

Where u = $\frac{dr}{dt}$ is the three dimensional velocity vector.

The square of the magnitude of the velocity four vector is given by

$$u_\mu \; u_\mu = \frac{u^2}{1-\frac{u^2}{c^2}} - \frac{c^2}{1-\frac{u^2}{c^2}} = -\, c^2$$

Which is Lorentz invariant.

5.9 Momentum four vector (p_μ):

The components of four – momentum p_μ are defined by

$$p_1 = m_0 \, u_1 = \frac{m_0 \, u_x}{\sqrt{1-\frac{u^2}{c^2}}}$$

$$= \text{m} \, u_x = p_x$$

$$p_2 = m_0 \, u_2 = \frac{m_0 \, u_y}{\sqrt{1-\frac{u^2}{c^2}}}$$

$$= \text{m} \, u_y = p_y$$

$$p_3 = m_0 \, u_3 = \frac{m_0 \, u_z}{\sqrt{1-\frac{u^2}{c^2}}}$$

$$= \text{m} \, u_z = p_z$$

$$p_4 = m_0 \, u_4 = \frac{m_0 \, ic}{\sqrt{1-\frac{u^2}{c^2}}}$$

$$= \text{m} \, ic = \text{i}\,\frac{E}{c}$$

Hence $p_\mu = (p_1, p_2, p_3, p_4) = (p_1, p_2, p_3, \text{m}\, ic) = (p, \text{i}\,\frac{E}{c})$ with p = mu

The square of the magnitude of the four – momentum is given by

$$p_\mu \, p_\mu \;=\; p^2 - \frac{E^2}{c^2} \;=\; -\,(E^2 - p^2 c^2)\,/\;c^2$$
$$= -\, {m_0}^2\, c^2$$

This p_μ is also called **energy – momentum** four – vector.

5.10 Acceleration four – vector (a_μ) :

Its components are defined by

$$a_1 \;=\; \frac{du_1}{d\tau} \;=\; \frac{du_1}{dt}\,\frac{dt}{d\tau} \;=\; \frac{d}{dt}\left(\frac{u_x}{\sqrt{1-\frac{u^2}{c^2}}}\right)\frac{1}{\sqrt{1-\frac{u^2}{c^2}}}$$

$$= \frac{1}{\sqrt{1-\frac{u^2}{c^2}}}\left(\;\frac{u_x}{\sqrt{1-\frac{u^2}{c^2}}} + \frac{u_x\, u\, \dot{u}}{c^2\,(1-\frac{u^2}{c^2})^{3/2}}\;\right)$$

But $\quad u^2 = {u_x}^2 + {u_y}^2 + {u_z}^2$ and

hence, $\quad u\,\dot{u} = u_x\,\dot{u}_x + u_y\,\dot{u}_y + u_z\,\dot{u}_z$

therefore,

$$a_1 \;=\; \frac{\dot{u}_x}{1-\frac{u^2}{c^2}} + \frac{u_x\,(u\,.\,\dot{u})}{c^2\,(1-\frac{u^2}{c^2})^2}$$

$$a_2 \;=\; \frac{\dot{u}_y}{1-\frac{u^2}{c^2}} + \frac{u_y\,(u\,.\,\dot{u})}{c^2\,(1-\frac{u^2}{c^2})^2}$$

$$a_3 \;=\; \frac{\dot{u}_z}{1-\frac{u^2}{c^2}} + \frac{u_z\,(u\,.\,\dot{u})}{c^2\,(1-\frac{u^2}{c^2})^2}$$

also
$$a_4 \;=\; \frac{du_4}{d\tau} \;=\; \frac{du_4}{dt}\,\frac{dt}{d\tau} \;=\; \frac{d}{dt}\left(\frac{ic}{\sqrt{1-\frac{u^2}{c^2}}}\right)\frac{1}{\sqrt{1-\frac{u^2}{c^2}}}$$

$$= \frac{i\,(u\,.\,\dot{u})}{c^1\,(1-\frac{u^2}{c^2})^2}$$

Thus
$$a_\mu \;=\; \left(\frac{a}{1-\frac{u^2}{c^2}} + \frac{u\,(u\,.\,a)}{c^2\,(1-\frac{u^2}{c^2})^2}\,,\quad \frac{i\,(u\,.\,a)}{c^1\,(1-\frac{u^2}{c^2})^2}\right)$$

$$a = \dot{u} = \dot{u}_x \hat{\imath} + \dot{u}_y \hat{\jmath} + \dot{u}_z \hat{k}$$

5.11 Force Four – vector(F_μ):

$$F\mu = \frac{dp_\mu}{d\tau} = \frac{d\ (m_0\, u_\mu)}{d\tau}$$

$$= m_0 \frac{d^2 x_\mu}{d\tau^2}$$

This equation called the Minkowski force equation and is presented in a form similar to Newton's equation.

In the limit u << c, the three – dimensional component are obtained as

$$F_k = \frac{dp_k}{d\tau}$$

$$= m_0 \frac{d^2 x_k}{d\tau^2}$$

Which is the classical Newton's equation.

The components of four – force $F\mu$ are

$$F_1 = \frac{dp_1}{d\tau} = \frac{dp_1}{dt}\frac{dt}{d\tau} = \frac{dp_x}{dt}\frac{1}{\sqrt{1-\frac{u^2}{c^2}}}$$

$$= \frac{F_x}{\sqrt{1-\frac{u^2}{c^2}}}$$

Similarly, $$F_2 = \frac{F_y}{\sqrt{1-\frac{u^2}{c^2}}}$$

$$F_3 = \frac{F_z}{\sqrt{1-\frac{u^2}{c^2}}}$$

$$F_4 = \frac{dp_4}{d\tau} = \frac{dp_4}{dt}\frac{dt}{d\tau} = \gamma \frac{d}{dt}\left(\frac{iE}{c}\right)$$

$$= \frac{i\gamma}{c}\frac{dE}{dt}$$

Thu $F_\mu = \left(\frac{F}{\sqrt{1-\frac{u^2}{c^2}}}, \frac{i\gamma}{c}\frac{dE}{dt} \right)$ ------------(1)

The four – force may be expressed in terms of four – acceleration vector as

$$F_\mu = \frac{dp_\mu}{d\tau} = \frac{d\ (m_0\, u_\mu)}{d\tau} = m_0\, a_\mu$$

Hence,

$$F_\mu = \left(\frac{m_0\, a}{1 - \frac{u^2}{c^2}} + \frac{m_0 u\,(u\,.\,a)}{c^2\,(1-\frac{u^2}{c^2})^2},\ \frac{i\, m_0\,(u\,.\,a)}{c^1\,(1-\frac{u^2}{c^2})^2} \right) \quad \text{-------------(2)}$$

Since equation(1) and (2) are the same, equating the space part of F_μ , we get

$$\mathrm{F} = \frac{m_0\, a}{(1 - \frac{u^2}{c^2})^{1/2}} + \frac{m_0 u\,(u\,.\,a)}{c^2\,(1-\frac{u^2}{c^2})^{3/2}} \quad \text{-----------------(3)}$$

$$\mathrm{F} = \mathrm{ma} + \frac{mu\,(u\,.\,a)}{c^2 - u^2} \quad \text{------------------(4)}$$

Where $\mathrm{F} = \frac{dp}{d\tau}$ is the three – dimensional force vector and in general is not equal to ma.

The fourth component of F_μ in (2) can be written as

$$\frac{i\, m_0\,(u\,.\,a)}{c^1\,(1-\frac{u^2}{c^2})^2} = \frac{i\gamma}{c}(\mathrm{F}\,.\,\mathrm{u}) \quad \text{-----------------(5)}$$

Because using (3)

$$\mathrm{F.u} = \frac{m_0\,(u.a)}{\sqrt{1 - \frac{u^2}{c^2}}} + \frac{m_0(u.u)\,(u\,.\,a)}{c^2\,(1-\frac{u^2}{c^2})^{3/2}}$$

$$= \frac{m_0\,(u.a)}{\sqrt{1 - \frac{u^2}{c^2}}}\left(1 + \frac{m_0\,(u\,.\,a)}{c^2 - u^2}\right)$$

$$= \frac{m_0\,(u.a)}{\sqrt{1 - \frac{u^2}{c^2}}}\ \frac{1}{1-\frac{u^2}{c^2}}$$

$$= \frac{m_0\,(u\,.\,a)}{(1-\frac{u^2}{c^2})^{3/2}}$$

Thus the fourth component of F_μ from (5) and (1) is

$$\frac{i\gamma}{c}\,\frac{dE}{dt} = \frac{i\gamma}{c}(\mathrm{F}\,.\,\mathrm{u})$$

$$\frac{dE}{dt} = \mathrm{F}\,.\,\mathrm{u}$$

The right hand side of above equation represents the power and the left hand side for a single particle $\frac{dE}{dt} = \frac{d}{dt}(m\, c^2)$ represent the rate of change of energy.

This is in accordance with the conservation of energy.

Thus the four – force F_μ is represented as

$$F_\mu = \left(\frac{F}{\sqrt{1-\frac{u^2}{c^2}}}, \quad \frac{i\,(F.u)}{c\sqrt{1-\frac{u^2}{c^2}}} \right) \qquad \text{--------------(6)}$$

The Minkowski force equation is

$$F_\mu = \frac{dp_\mu}{d\tau} = m_0 \frac{du_\mu}{d\tau} \qquad \text{----------------(7)}$$

It represent the fundamental equations of mechanics in the covariant four – vector form with the components given by equation (6)